THE NATURE OF MATHEMATICAL PROOF*

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1. Introduction. In presuming to come before you with such a title as "The nature of mathematical proof," let me assure you that I am not doing so with the idea of presenting any new or startling facts. I do this because I think it is good for us, as mathematical specialists of one sort or another, speaking in terminologies that frequently render us obscure even to one another, to pause and reflect now and then on just what we are doing and how we are doing it. For certainly we put a great part of our time and energy into the act of proof. We ask one another, "Do you think anyone will ever prove the Fermat Theorem?" or, "Do you think anyone will ever prove the continuum hypothesis?" A host of mathematical questions would receive answers if only we were able to find proofs for the theorems which hold the keys to their solutions.

In our teaching of such seasoned parts of mathematics as classical geometry, we continually ask our students to prove theorems. We see frequently in our journals articles which embody only new proofs of old theorems, which are justified either by increased simplicity or by the requirement of fewer logical hypotheses. And we have observed the study of proof methods which has now gone on for many years in mathematical logic, particularly among those who have been styled the Hilbert school of thought, or have come under its influence. Also we are aware that proofs have been published which have not found general acceptance, not because any error was made in the logical reasoning, but because some principle which may have theretofore been found quite innocuous, or, indeed, passed unnoticed, is suddenly revealed in all its power. This is particularly the case when the principle renders possible the proof of some theorem that a respectable body of mathematicians have already decided should not be true.

I suppose there is considerable ground for the logistic thesis as originally stated by Russell, to wit, that "Pure mathematics is the class of all propositions of the form 'p implies q,' where p and q are propositions containing one or more variables, the same in the two propositions, and neither p nor q contains any constants except logical constants." One of the chief of these constants is implication. And Russell asserts that in addition to these logical constants, mathematics uses a notion which is not a constituent of the propositions which it considers, namely the notion of *truth*.† If there is any more prominent notion in the popular mind concerning the nature of mathematics, other than that it deals with "figures," it is probably that the results of mathematics are certain and absolutely true.

* Except for the introduction of a series of headings, this is an address delivered by invitation to the meeting of the Association at Chicago on November 28, 1943. At the suggestion of the editor, an appendix is attached embodying a brief exposition of some of the concepts employed; readers unfamiliar with the latter may wish to read this appendix first.

† Russell, B., The Principles of Mathematics, vol. 1, Cambridge, 1903. For a critique by Russell himself of the material quoted above, see the Introduction to the Second Edition, New York, 1938.

Now it is chiefly the following aspects of mathematics that I should like to discuss: (1) The nature of this process or procedure that we call "proof," and (2) the relation of the end result of the process to "truth." Specifically, what position does proof occupy in mathematics, and how does it influence the so-called truth or certainty of our results?

2. Types of proof. I cannot attempt here any complete or exhaustive analysis of proof, even if I felt competent—for this you may consult the works, either classical or current, of Russell and his followers, or of Hilbert and his followers, for example, according to your philosophical inclinations. As a matter of fact, I am using the word "proof" in a very broad sense, not in a narrow technical sense. For my purposes I want merely to recall some of the general outlines of proof as we apply it every day both in class and research. I am not interested in all the step-by-step processes, as, for example, substitution, which go to make up the atomic elements of proof.

2a. Mathematical induction; transfinite induction. One element which I cannot avoid, since it not only develops, like the amoeba, a long chain of descendants, but itself occupies a central position among our proof outlines, is the well-known syllogistic type of proof. Given p, and that p implies q, then q follows. From a certain point of view, our basic proof by mathematical induction seems to be nothing but an omega series of syllogisms, as pointed out by Poincaré:* The theorem is true of the number 1. Now if it is true of 1, it is true of 2. Hence it is true of the number 2. And so we proceed. But of course we don't "proceed" in reality—what we do, in order to avoid an infinite and unending series of syllogisms, is to use a property of the natural number sequence to knock over a row of syllogisms much as a child tips over a row of vertical blocks by setting them in the proper juxtapositions and then tipping over the first of them. Everything depends on having the proper juxtapositions, and it is a fundamental property of the sequence of natural numbers, as formulated for instance by Peano,† that the order type guarantees the mathematical induction process. In other words, we have introduced an infinite totality, and by properly ordering its terms, succeeded in applying the syllogism only once to prove a theorem about the infinite totality.

Modern mathematics has incorporated another extension of this mode of proof, namely, proof by transfinite induction. From the fact that the omega series is only an elementary type of well-ordered series, we generalize to the application of the induction method to any well-ordered sequence. All we need is that after having considered any part of the sequence, there be a first element in what is left, provided the latter is not empty. And in addition to the fact that if true for an element then true for its successor, the fact that if true for all predecessors of an element, then true for the element. For example, if for a given well-ordered set of elements $\{x_\alpha\}$ we have defined a function $f(x_1)$, and have

* Poincaré, H., La Science et l'Hypothèse, Paris, 1912, Chap. I, §V.

† Peano, G., Formulaire de Mathematiques, vol. II, Turin, 1897. See §2, p. 1.

defined $f(x_\beta)$ whenever $f(x_\alpha)$ for $\alpha<\beta$ has been defined, then we can assert that $f(x)$ is defined for *all* x_α of the sequence.

But notice the difference in application from that of ordinary mathematical induction. In the latter case the sequence x_α is ordinally similar to the natural number sequence, $1, 2, \cdots, n, \cdots$, and is probably hypothesized to begin with, as in the theorem that for every n in the natural number sequence, $1+2+\cdots+n=n(n+1)/2$. And the sequence of natural numbers is unquestionably the most reputable of infinite totalities. But transfinite induction, on the other hand, is applied to infinite totalities which for various reasons are not in such favor, and which frequently are only timidly mentioned by those who do favor them, as for example in the case of the set of all ordinals of the second class. And, moreover, the arrangement of the totality in a well-ordered sequence for purposes of induction is usually imposed at some step in the process of proof.

We find, for instance, an eminent contemporary avoiding a proof which involves the notion of the totality of all the ordinal numbers of the second class, with the explanation: "I regard the intuition of the totality of transfinite numbers of the second class as defective, and I believe that there is a certain degree of illusion in the clarity which the usage of the notations for the 'small' transfinite numbers of the second class casts on our intuition of this totality. We are led thus to inexact intuitions, and this fact as well as other analogous facts compel us to be very cautious in the introduction of transfinite numbers of the second class by the logical path."* And on the part of those who do not take so radical a view we find instance after instance, as in the case of a well-known publication of Kuratowski,† of papers devoted either partially or completely to the devising of means of eliminating transfinite numbers from mathematical proof. And while many of these latter are careful to include words of explanation to the effect that of course they understand that transfinite numbers are only an extension of the arithmetic of integers, they are plainly uneasy about the "existence" of these numbers, and hazy about where to draw the line of compromise between elegance of proof and elimination of these numbers. It seems rather clear that *in order to have mathematics we must have the infinite; but just HOW infinite, that is the question*! To incorporate into our logical apparatus of proof the infinite totality of natural numbers is quite proper, but to assume the existence (whatever that may mean) of the infinite totality of second class ordinals is to risk rejection of one's proof by part of the mathematical public, as well as an invitation to a still larger part of that public to try to find a proof which makes no use of transfinite numbers—or at least not *all* the numbers of the second class.

2b. Proof by example. Let me next mention proof by example. We use it both constructively and destructively. The usual constructive use to which we

* Lusin, N., Leçons sur les Ensembles Analytiques, Paris, 1930, p. 27.

† Kuratowski, C., Une méthode d'élimination des nombres transfinis des raissonements mathématiques, Fundamenta Mathematicae, vol. 3, 1922, pp. 76–108; in footnote 1, citations will be found to other papers of a similar nature.

put it is that of transferring the question of the consistency of a set of postulates to some other domain, as Hilbert did when he reduced the question of the consistency of his geometry postulates to that of the arithmetic of real numbers. In the case of postulate systems whose generality allows of finite interpretation in the world of our senses, we usually argue that the law of contradiction disallows the existence of contradictory propositions—but the method is generally admitted to be no guarantee of consistency when the degree of infinitude demanded by the system drives us to the use of strictly mental concepts—as, for example, transfinite ordinals. The destructive use of the method of proof by example is that of proving the impossibility of proving a theorem by furnishing an "exception to the rule." Thus, to our forebears, it seemed to be the rule that all continuous functions defined over an interval had a derivative at least somewhere, until Weierstrass gave his famous example. We continually settle problems by this method—perhaps the famous four-color problem will be solved this way. But, of course, we are here again faced with the question as to the validity of an example, and the distressing lengths to which some authors go in giving a rigorous formulation of an example are no doubt due in good part to the desire to leave no question open as to its validity. Unfortunately the existence question rears its head again, if the example reposes in the domain of the not generally accepted; but at least in this case the example seems to have a more general effect, inasmuch as the scoffers don't usually resume the search for the proof of the theorem whose impossibility has apparently been proved within domains or by methods which they reject. I haven't heard lately of anyone trying to prove that every bounded real function is measurable!

2c. Reductio ad absurdum. Let us get on to a much more popular type of proof—that of reductio ad absurdum. In good repute among many formalists and logicians both for its elegance and its incisive character, its use has been limited by those of an intuitionist leaning because of its dependence on the law of the excluded middle. Of course, in one respect, namely, in proofs of nonexistence, its use is approved by all, no matter of what school of thought. If the assumption of the existence of a mathematical entity leads to contradiction, no one wants to maintain its existence. But one notes in recent years an increasing use of various constructive methods for existence proofs. To ask for proof of existence by construction is not sufficient basis to label one an intuitionist, since many have come to feel that unless the proof of existence by reductio ad absurdum offers very great advantages by reason of its simplicity or avoidance of complicated construction, there is not sufficient justification for its use; it is better to know how to build up the entity desired than to show, for example, that the class of which it is an element is not empty. It is somewhat amusing, I think, that some of us become so used to the "contra-positive" type of argument that we automatically take an argument that is ideally suited to the constructive form of proof, and twist it into a reductio ad absurdum. For example, consider the classic theorem that there exists an infinity of prime num-

bers. As you know, Euclid showed how, given a certain finite set of prime numbers, one can construct a prime number not in the set, and it is no task to go a step further and make this number unique. Hence it seems hardly necessary to argue that if it be assumed that the set of all prime numbers is finite, say k in number, then by Euclid's method there can be shown to exist a $(k+1)$th prime and hence a contradiction, inasmuch as Euclid's method itself provides a valid procedure for constructing an infinite series of primes by recursion (although I am aware I am using the word "constructive" here in a sense which many may not admit—about which more presently). Of course, whether one wants to give up the extremely elegant proofs of the fundamental theorem of algebra which use the contra-positive method, to replace them by the intuitionistic method of proof* is another question, and one that I presume can only be decided by an appeal to one's innate convictions on the foundations.

2d. Constructive methods. Let us look more closely at the constructive method, however. What do we mean by construction? I presume that one would like to stipulate that a proof by construction should, theoretically at least, enable the one who cares to carry through the steps of the proof in any particular case to construct the entity whose existence is proved. Suppose, for example, that this entity is a real number. Do we mean that the proof is to enable us to start writing down the number, digit by digit, so that independent workers, by following the steps provided in the proof, would generate the same series of digits no matter how far they carried the process? For instance, by employing the Cantor diagonal process on the well-known ordering of the algebraic numbers in an omega series, one can constructively or effectively define a unique transcendental number, corresponding not only to every natural number n, but to every transfinite ordinal of the second class which may be constructively defined in a certain sense into which I will not go here; and, indeed, in such a way that if one specifies, for example, the transfinite ordinal ω^2, one can start writing down within a few minutes the corresponding transcendental number, digit by digit.

But the trouble with this definition of constructive proof of existence of real numbers is that one can easily give definitions that do not furnish a means to start writing out a series of digits, but that plainly do fix a certain real number. For example, consider the number N which is equal to

$$1 + (-1)^{\rho}/(10)^{\rho},$$

where ρ is the number of the first decimal place in the decimal development of π where a sequence of digits 0123456789 commences; or, if there is no such number ρ, then $N=1$. If we know that ρ exists and is even, we can begin to write the number $1.0000 \cdots$ until we get to the ρth place, where there is a 1. But if ρ is odd, then we must begin to write $0.999 \cdots$ (to ρ digits). But we

* See Brouwer, L. E. J., and de Loor, B., Intuitionistischen Beweis des Fundamentalsatzes der Algebra, K. Akad. der Wetenschappen, Proceedings of the Section of Science, vol. 27, 1924, pp. 186–188.

don't know whether ρ exists or not, and in the absence of such information we can't even begin to write down the proper digits for N. Nevertheless we have to admit that the definition fixes a number N, and we notice that it fixes it to any required degree of approximation, even though we can't start writing down its digits. Evidently, then, we may have to decide that existence of a real number is proved constructively when for any given positive number ϵ the number is determinate within an error of at most ϵ.

But granted that this definition is satisfactory, we run into another difficulty: Are we going to have to set up a new criterion for what we mean by proof by construction every time we prove the existence of a new mathematical entity? That is, is our definition to be, in a sense, a function of the thing whose existence is to be proved? As another example, consider the Borel Theorem for a finite interval of the real number system. Let G denote a set of open intervals covering a closed interval ab. We may define what would certainly seem to be a constructive method for selecting a finite set of intervals of G which cover ab. All we need do is first find that point, call it a_1, which is the least upper bound of the points x such that both a and x are simultaneously covered by an interval of the collection G.* Then find that point, which we call a_2, which is the least upper bound of the points x_1 such that both a_1 and x_1 are simultaneously covered by an interval of the collection G. And so on. A non-existence argument by the contra-positive method is easily given to show that we finally reach an a_k (k a natural number) which is the first to exceed b. Then to get the finite set of intervals of G covering ab we first select an interval of G, say I_k, which covers a_{k-1} and b; this is possible because of the definition of a_k. Then we select an interval I_{k-1} which covers a_{k-2} and overlaps with I_k—this is possible because of the definition of a_{k-1}. Then we select an interval I_{k-2} of G which covers a_{k-2} and overlaps with I_{k-1}, and so on. The set of intervals $I_1, \cdots, I_k$, where I_1 is finally selected to cover a and overlap with I_2, is a finite set of intervals of G covering ab.

I said that this would seem to be a constructive proof. But I am aware that although only a finite number of selections of intervals of G are required, two independent workers, given the same data concerning the collection G, could hardly be expected to produce the *same* covering of ab by intervals of G. So if this is to be a constructive definition of a covering, another definition of what we mean by constructive proof is needed. But in providing such a definition, we would again be making our definition of constructive proof a function of the sort of thing whose existence is to be proved. And finally we may conclude that we cannot say in general just what we do mean by constructive proof, except that for a real number it may mean one thing, for another mathematical entity it may mean another thing, etc.†

* We assume here that the intervals of G are actually *given* (analytically defined), in such a way as to enable us to find this number, a_1. Interesting examples of this sort may be given.

† See also §8 of the article by K. Menger, Some applications of point-set methods, Annals of Mathematics, vol. 32, 1931, pp. 739–760.

2e. Principles not universally accepted. Of course, the fact that a proof of existence does not use the reductio ad absurdum argument is no guarantee that constructive methods have been employed. This is particularly the case when use is made of the Zermelo choice axiom or the continuum hypothesis.* And it is admitted that the existence of many mathematical entities cannot be proved without the use of one or the other, or both of these principles. And there are many among us, not all intuitionists, who refuse to recognize such proofs. I have noticed particularly that those who don't *need* such aids in their research are quite prone to pass laws against the use of such powerful tools as the choice axiom, the continuum hypothesis, transfinite induction, *etc.*—the number-theorists, for example (shades of Kronecker!). Be it to Brouwer's everlasting credit that he was not a number-theorist!

On the other hand, I think it is safe to say that even the most formal of the formalists considers it good ethics to post a sign somewhere if, for example, he uses the Zermelo choice axiom in a proof. Topologists know of one book† in which the author's use of the choice axiom is to be so frequent that he just puts the sign in the introduction and then proceeds to use the axiom freely—sort of a blanket license. And in Sierpinski's books‡ one finds classifications of theorems according to whether their proofs depend on the choice axiom, the continuum hypothesis, etc. And many an editor who had not the least tinge of intuitionism has accepted for publication an article giving the first proof of a reputable theorem without the use of the choice axiom or the continuum hypothesis. Most of us are aware of theorems that we would like to see proved without the use of these hypotheses, if only for esthetic reasons. But the *demand* for such proofs before admission to mathematical distinction is frequently unreasonable, I feel, in view of the very indefiniteness or scantiness of the hypothesis with which one has to work. Indeed, it seems clear that if we are going to have certian parts of mathematics at all, we shall have to have such principles. Certain mathematical problems are simply unsolvable without the continuum hypothesis. One may, like the intuitionist, solve such problems by the alternate procedure of tossing them out of the realm of mathematics—a cure by extermination, as it were—but most of us dislike to go to such lengths.

But I think that for my purposes I have covered enough of the materials that enter into proof in mathematics.

3. Mathematical dogmatism. Now we are probably quite familiar with the fact that mathematical proof is a function of the time. History shows this conclusively—Euclid would probably have complained of the lack of rigor displayed by his predecessors; Weierstrass felt it necessary to reorganize the foundations

* Cf. Sierpinski, W., Les exemples effectifs et l'axiome du choix, Fundamenta Mathematicae, vol. 2, 1921, pp. 112–118.

† Moore, R. L., Foundations of point set theory, Amer. Math. Soc. Coll. Pub., vol. XIII, New York, 1932.

‡ Lecons sur les nombres transfinis, Paris, 1928 (Chaps. VI and XII); Hypothèse du continu, Warsaw, 1934.

of analysis; and so on. The present is a time when it seems appropriate to reflect on the new and still uncertain elements that have come into mathematical proof. One may berate those who make their proof methods dependent on some particular mathematical philosophy, such as intuitionism, but at the same time be taking considerable license when, in order to prove a much desired result, he resorts to methods that are both novel and uncertain. We may, if we like, hide behind the excuse that a proof which uses a hypothesis which so far as anyone knows has never led to contradiction, is at least a clue to a possible mathematical fact, and as such it may encourage us to establish the fact on firmer ground. This recalls Weyl's observation to the effect that giving a non-constructive existence proof is like informing the world that somewhere there exists buried treasure but not stating where it lies!* But more to the point is the fact that there is certainly some worth in a proof which shows that the theorem is in the same category with the not generally accepted hypothesis, as, for example, when it is shown that certain theorems are equivalent to the continuum hypothesis, or just as it was shown that the choice axiom, well-ordering theorem and comparability principles are all equivalent. Nevertheless, these facts are no ground for dogmatism.

The present division of mathematical thought into schools, insofar as it degenerates into dogmatism, is not, in my opinion, healthy. On the other hand, insofar as it leads to dispassionate discussion of fundamental principles, it is a decidedly healthy development. I cannot refrain, in this connection, from expressing my protest against what I like to call, variously, the "mathematical dogmatist" or "the mathematical fascist." He may have no religious philosophy or affiliation in the ordinary sense, yet if you venture to doubt the validity of some one of his favorite proof-methods, you may find yourself in danger of physical violence. In his eyes, you are a mathematical free-thinker, an anarchist, and paradoxically although he may be both of these in his political beliefs, he won't tolerate them in mathematics. It is as though he were the mathematical prototype of the arch-reactionary whose political and economic emotions stem from his possession of an unusually large share of the world's goods; the mathematical reactionary is motivated by his possession of a large body of mathematical theory whose foundations are in danger of attack by the mathematical revolutionist.

In the case of some individuals, it is almost as though mathematics had become a kind of religious fanaticism, rather than a labor of love. They give one a feeling that we have in mathematics the intuitionist, formalist and logistic "Theologies." Insofar as the leaders of these systems become dogmatic, just so far do they become the mathematical analogues of the Dalai Lama or the Pope of Rome. And although I am not concerned here with dogmatism in religion, I think I am justified in protesting its presence in mathematics. Of course, analogy must not be pushed too far—we don't venture to predict the creation

* Weyl, H., Philosophie der Mathematik und Naturwissenschaft (Teil I), Handbuch der Philosophie, Abt. IIA, Munich and Berlin, 1926, p. 41.

of an annual collection to be called "Kronecker's pence" to be presented to Brouwer. But the claim to possess mathematical infallibility, the attempt to set up formalistic or intuitionistic proof rituals—any of these acts and their like, are not, I venture to say, exemplifications of the truly mathematical spirit. On the other hand, to set forth these ideas as suggestions for the cure of mathematical ills or the improvement of proof-methods in the direction of greater rigor, is, it seems to me, to proceed in a truly scientific way.

As mathematicians, dealing with a subject that must be kept scrupulously abstract, which has no material connections except in the manner of what we call "applied mathematics," we must ever be on guard against dogmatism. It is natural for the layman to regard our works with awe, to think of us as the possessors of absolute truth, since we have been singularly successful in avoiding contradiction in applied mathematics. But we must not allow this veneration with which we are regarded to tempt us to set up our own mathematical cults or political philosophies. Rather we need to practice democracy, not sneering at new ideas as dangerous to our pet mathematical theories, but regarding them as possible improvements on the existing system. I think that the worth of new ideas can safely be left, in the long run, to the judgment of the mathematical electorate. A mathematical idea that never "takes" is probably not worth taking—although we must remember that here, as in other human activities, the long range point of view should prevail. What is unpopular today may become the fashion of tomorrow—which is only further admonition to be receptive to new ideas.

And I want to point out here what seems to me to be a heartening fact, although I realize that I may be sorely trying some of my listeners, personal philosophies: Namely, that no matter if we do use, frequently, questionable methods of proof, or even make outright errors in proof, we are usually gathering mathematical fruit. That the calculus of Newton and Leibnitz had virtually no basis at all from the viewpoint of modern standards did not invalidate the calculus as mathematics. All it needed was bolstering up. Even now we argue about such items as Duhamel's Theorem or what we mean by the differential, but no one suggests that we throw the calculus out of our mathematical libraries, for we recognize intuitively that it is a body of acceptable mathematics—still susceptible to improvement, perhaps. It is not that we consider that the theorems of the calculus have been elegantly and conclusively proved in the theory of functions. To put the matter bluntly, it is a case of our ***knowing mathematics when we see it.*** And we don't set out to prove a theorem in the first place unless ***we think it is worth proving.***

4. Source of the theorem to be proved. And this brings me to a consideration of the source of the thing we set out to prove—the theorem. Where do we get it? I think most of us would say from our intuition. Of course, we do not use the word "intuition" here in the sense of the intuitionist—we mean that intuition which guides the selections of the mathematical electorate. As has often been remarked, the Intuitionist (with a capital *I*) seems to use this intuition to tear

down part of the mathematical edifice—whereas here I use the term more in accord with its usual philosophical meaning. It is the sense in which most of us who engage in mathematical research understand the term—that mental medium that generates the mathematical hunch. What often seems to us perhaps the great good fortune or luck of those analysts of an earlier day whose ignorance of convergence criteria did not prevent their constructing mathematics of probably eternal significance was, I think, *mathematical intuition*, not luck. Only we must again beware of dogmatism or professionalism. As every research mathematician knows, the individual intuition is an amiable and usually reliable guide, but on the other hand it is evidently only human and prone to error. It is the collective intuition that seems more to be relied upon, and we may, I believe, have faith in its long-run judgment as to what is valid mathematics. And this is as it should be, since it is the mathematical intuition that makes mathematics. Without it, we would have nothing to prove. I am quite willing to grant that, with proof or without proof, or with proof by what some of my contemporaries insist is no proof at all, the mathematical theorem is not necessarily a statement of fact or truth in the ordinary sense. But it may be a decidedly elegant and artistic piece of mathematical statement. And as such it may be adopted by the mathematical electorate into the body of mathematical literature.

5. The role of proof. Now, granted that the mathematical theorem comes from the intuition, what is the role of proof? It seems to me to be only *a testing process that we apply to these suggestions of our intuition*. We have various kinds of tests, some of them, like the syllogism, substitution, or finite selection, universally relied upon, but others, such as those making use of the choice axiom or the total second ordinal class, not generally admitted. But even though some of these types of proof fail of general acceptance, is there any reason why we should not take successful passing of such tests as evidence that the concept is a possible candidate for admission to the body of mathematical theory? Some of my contemporaries have even gone so far as to question the sanity of any individual who makes so bold as to entertain thoughts about the continuum hypothesis. I am reminded that an eminent contemporary, in an address before this Association not so long ago*—an address which I take great delight in reading every now and then because of its fresh wit and clarity—took occasion to deride the statement which Sierpinski makes about the continuum hypothesis in the introduction to his book to the effect that "There are some people, even among eminent scholars, who doubt the possibility of ever solving the problem of the continuum. Under these circumstances, the consequences of the hypothesis of the continuum can be considered practically as if they were true." Now, although I don't like the occurrence of the word "true" here, if the assumption of a principle, as say of the hypothesis of the continuum, leads to no contradiction, why not take it as valid mathematics? I don't say that we should make

* Bell, E. T., The place of rigor in mathematics, this MONTHLY, vol. 41, 1934, pp. 599–607.

absence of contradiction the sole criterion for mathematical admissibility, but absence of contradiction together with mathematical usefulness are, or ought to be, sufficient. And certainly no one can deny that the continuum hypothesis is mathematically fruitful. I note, parenthetically, that Gödel established its consistency with certain well-known axioms of set theory provided the latter are consistent—in fact he did much more than this in that he included the axiom of choice and the generalized continuum hypothesis.* One of our most orthodox methods of proof is that of reductio ad absurdum, and if the assumption of a hypothesis leads to no contradiction, why not admit this as evidence for its mathematical admissibility? And it seems to me that the continuum hypothesis has passed plenty of tests, aside from the result of Gödel which I have just mentioned, in its mathematical fruitfulness.

6. Conclusion. In conclusion, then, I wish to repeat my belief that what we call "proof" in mathematics is nothing but a testing of the products of our intuition. Obviously we don't possess, and probably will never possess, any standard of proof that is independent of time, the thing to be proved, or the person or school of thought using it. And under these conditions, the sensible thing to do seems to be to admit that there is no such thing, generally, as absolute truth in mathematics, whatever the public may think. Our intuition suggests certain results, and they seem mathematically desirable—and, moreover, prove to be generally liked by the mathematical public. We test them by what we call a proof, of course—if they didn't pass any such test, or, worse still, if their assumption leads to contradiction, we wouldn't pass them on to our colleagues for their judgment as to their worth. Naturally, we cannot assume a dogmatic attitude about them—we shouldn't, as far as that is concerned, be dogmatic even about arithmetic—but we can rely, I think, upon the combined evidence of our intuition and the test which we call "proof," even though the latter may be rejected by some of our colleagues who have somehow attained, possibly from psychic sources, a preferred position among the gods that they think rule over the mathematical universe.

I suppose I am an intuitionist in the sense that I think that a mathematician is to be judged by the quality and reliability of his intuition at least as much as by his ability to prove something. And I am inclined to agree with the statement which I have heard attributed to the late E. H. Moore, that "Sufficient unto the day is the rigor thereof." We certainly owe a lot to those mathematicians of an earlier day whose rigor was rather non-existent from present-day points of view. And if you'll pardon a personal interpolation, if my intuition tells me that a certain theorem is desirable, and seems free of contradiction, I'll prove it in any manner I best can; if my proof makes use of methods or assumptions that certain more dogmatic of my brethren choose to reject outright, I'll prove it just the same and hopefully present it to a reputable journal—

* Gödel, K., The consistency of the continuum hypothesis, Annals of Mathematics Studies, No. 3, Princeton, 1940.

especially if it seems mathematically important. At the same time, however, I won't talk about "truth," "absolute rigor," and their ilk—for all I know they are will-o'-the-wisps—ideal conceptions, possibly, but with no natural habitat in the mathematical world.

APPENDIX*

I. Totalities. We use the term *totality* in Section 2a to indicate a collection of things supposed already existent. Thus, the collection, N, of natural numbers (positive integers) is not a totality when conceived of as in some sense a variable finite collection beginning with 1, 2, 3 and increased at will by the addition of a natural number obtained, say, by adding 1 to the largest number already in the collection. Rather, the totality N contains *all* natural numbers—we recognize that 123,341,687,912,847,712,868 is an element of this totality although perhaps it was never before written down.

In like manner we speak of the totality R of real numbers. In this case each element of the totality—a real number—is itself a totality, say, an ordered totality of digits in the decimal system. Thus $\sqrt{2}$ is $1.414\cdots$, each digit in the array being determined by a certain law (*e.g.*, the algorithm for finding the square root). We cannot expect, however, to know such a law for the determination of the digits of every real number.

Between totalities there can be set up an equivalence relation, namely, the relation of 1-1 *correspondence*: If A and B are two totalities and if there exists a function f on A such that for every element a of A, $f(a)=b$ is an element of B and such that (1) if $a_1 \neq a_2$, $f(a_1) \neq f(a_2)$, (2) for every element b of B there is an element a such that $f(a)=b$, then f is called a 1-1 correspondence between A and B,† and A and B are said to *have the same cardinal number*; in symbols, $A \sim B$. Clearly (1) $A \sim A$, (2) $A \sim B$ implies $B \sim A$, and (3) $A \sim B$, $B \sim C$ imply $A \sim C$. Consequently each totality A determines a class $\mathfrak{C}(A)$ consisting of all those totalities X such that $A \sim X$, and if $A \sim X$ then $\mathfrak{C}(X)=\mathfrak{C}(A)$.‡ We may, with Frege, use the concept of the class $\mathfrak{C}(X)$ to define number. From this point of view, if A is the totality of all states in the United States, then $\mathfrak{C}(A)$ is customarily written 48. We call $\mathfrak{C}(A)$ the *cardinal number* of A. The cardinal numbers of infinite totalities are called *transfinite*. The number $\mathfrak{C}(N)$ is denoted by $\aleph_0$ (aleph-null), and any totality T such that a 1-1 correspondence exists between N and T has consequently the cardinal number $\mathfrak{C}(T)=\aleph_0$. We also specify that $\mathfrak{C}(T)=\aleph_0$ by the statement that "T is denumerable." And if a set T is denumerable, its elements can be considered as forming a sequence $t_1, t_2, \cdots, t_n, \cdots$.

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* For more extended expositions of concepts that are of necessity but briefly described here, the reader is referred to the excellent books of Sierpinski cited above.

† It would be more appropriate to say "1-1 correspondence between the elements of A and the elements of B," but we shall use the abbreviated form despite its inexactness.

‡ We use $=$ here to denote identity.

The cardinal number $\mathfrak{C}(R)$ is denoted by c. It will be shown below that $\aleph_0$ and c are different. An elementary statement of the *continuum hypothesis*, which is first referred to in Section 1 above, is that if M is a totality of real numbers that is not denumerable, then the cardinal number of M must be c.

N.B. Although the word "totality" seems to express most exactly the ideas involved, we frequently use the more common and less cumbersome synonymous term "set" in what follows.

II. Ordered totalities. The totality N is an example of an *ordered* infinite totality, where by "order" we mean the natural order $1<2$, $2<3$, etc.—"order by magnitude." Similarly, R is an ordered totality, again order by magnitude in the usual sense. But whereas in this case certain parts of R—such as the totality of positive real numbers—have no smallest element, every part of N has a smallest element. The distinction is denoted by the term "well-ordered," N being well-ordered and R not well-ordered. Explicitly, a totality T is *well-ordered* with respect to a binary relation $<$ if (1) T is simply ordered (*i.e.*, if x, y, z denote elements of T, then (a) if x and y are not identical, $x<y$ or $y<x$, (b) if $x<y$ then x and y are not identical and (c) $x<y$, $y<z$ imply $x<z$), and (2) if T' is a part of T (possibly T itself), then T' has a first element.

Between well-ordered totalities there can be set up a special type of equivalence relation $\approx$: If W_1 and W_2 are well-ordered sets and there exists a 1-1 correspondence between them which preserves order (*i.e.*, if a, b are elements of W_1 and $a<b$ then $f(a)<f(b)$ in W_2, and conversely), then we say W_1 and W_2 *have the same ordinal number*, and write $W_1 \approx W_2$. And just as we defined cardinal number above we may define for any set A which is well-ordered an *ordinal number* $\mathfrak{O}(A)$, which is the class of all well-ordered sets X such that $A \approx X$. But it should be emphasized that whereas $\mathfrak{C}(A)$ is unique for a given set A, $\mathfrak{O}(A)$ is unique only for a given well-ordering of A. For this reason it is advisable to replace $\mathfrak{O}(A)$ by $\mathfrak{O}(A, <)$, where $<$ is a symbol indicating the given ordering. For example, consider (1) N and the order $<$ given by its natural ordering, and (2) N and the order $<'$ defined as follows: If a, b are both odd, or both even, then $<'$ is the same as $<$; but if a is odd and b is even, then $a<'b$ (thus $2<3$ but $3<'2$). Then N is well-ordered with respect to both $<$ and $<'$, but it is easily shown that $\mathfrak{O}(N, <)$ and $\mathfrak{O}(N, <')$ are different. Generally, then, the same set has many different well-orderings, and to each of these corresponds an ordinal number.

In particular, the ordinal numbers that may be obtained from the various well-orderings of N are called the *ordinal numbers of the second class*—thus $\mathfrak{O}(N, <)$ and $\mathfrak{O}(N, <')$ above are two of the ordinals of the second class. The number $\mathfrak{O}(N, <)$ is denoted by ω, and is called an "omega series." Note, however, that the ordinal number of a finite set is unique—there is only one type of well-ordering for a set of 3 elements, for example—and it is customary to lump all such finite ordinals into one class, called the *first class*. And ordinal numbers not in the first class are called *transfinite ordinals*.

III. The well-ordering theorem, choice axiom, comparability. It is natural to inquire into the possibility of well-ordering a totality which, as it comes to our attention, is either not ordered at all, or if ordered is not well-ordered (for example, R is simply ordered with respect to its order by magnitude, but not well-ordered). As stated in II, $\aleph_0 \neq c$. It will be noted that this is equivalent to asserting that there is no well-ordering $<''$ of R such that $\mathfrak{O}(R, <'') = \mathfrak{O}(N, <)$. The classical proof of this assertion is that of Cantor which (with minor modifications) is as follows:

Assume $\aleph_0 = c$, and therefore that R can be considered as ordered in an omega series $r_1, r_2, \cdots, r_n, \cdots$. Let us denote the decimal part of each r_n by $a_1^n a_2^n \cdots a_n^n \cdots$, and let r be the number $0.b_1 b_2 \cdots b_n \cdots$ where b_n is 1 in case a_n^n is not 1, and b_n is 2 if a_n^n is 1. Then r is recognized by its form to be a real number, yet differs from r_1 in the first decimal digit, and generally from r_n in the nth decimal digit—contradicting the fact that all elements of R were supposed to be in the series $r_1, r_2, \cdots, r_n, \cdots$.

Now no one has ever determined a well-ordering for R. Moreover, no one has ever determined all types of well-orderings even of the totality N. Thus, although we know the $\mathfrak{O}(N, <)$ and $\mathfrak{O}(N, <')$ above, as well as infinitely many other ordinal numbers derived from the well-orderings of N, it is fairly well agreed that it would be impossible to define all such—in other words, to define effectively all transfinite ordinals of the second class. It is not surprising, then, that a well-ordering of R, which in a sense would exhibit an "infinitely longer" sequence than any well-ordering of N, has never been found. And it is also not surprising that many mathematicians felt that no well-ordering of R could possibly exist.

Despite this, and despite the fact that one can demonstrate formally the existence of infinitely many other cardinal numbers greater, in a sense that we do not define here, than either $\aleph_0$ or c, Zermelo proved in his famous paper *Beweis, dass jede Menge wohlgeordnet werden kann*,* that every totality has a well-ordering; that is, given any set A whatsoever, there exists an order relation $<_a$ such that A is well-ordered with respect to $<_a$. A close inspection of the proof reveals, as it did to Borel,† that the proof is based upon a principle that had never received strict formulation and yet had undoubtedly been employed time and again, the principle now known variously as the "choice axiom" or "Zermelo postulate." Suppose M is a set; let us designate some particular element m of M as a *representative element* of M and write $r(M) = m$. (For example, if M is the set N, we might arbitrarily let $r(N) = 6$. Or if M is the set of residents of Maine, we can select any particular resident of Maine as $r(M)$.) The simplest form of the choice axiom is as follows: If $\mathfrak{M}$ is a collection of mutually exclusive sets M, then there exists a *representative set* $\mathfrak{R}(\mathfrak{M})$ whose elements are representative elements $r(M)$, there being exactly one such rep-

* Mathematische Annalen, vol. 59, 1904, pp. 514–516.

† Borel, E., Quelques remarques sur les principes de la theorie des ensembles, Mathematische Annalen, vol. 60, 1905, pp. 194–195.

resentative for each set M in $\mathfrak{M}$. Thus if $\mathfrak{M}$ is the set of all states in the United States, where each state is considered as the collection of all its residents, it being assumed that no person is a resident of more than one state, then $\mathfrak{R}(\mathfrak{M})$ would consist of 48 people. The point of the "axiom" is that we can assume that it is not necessary actually to *name* each person in $\mathfrak{R}(\mathfrak{M})$; we can just assume that such a set as $\mathfrak{R}(\mathfrak{M})$ exists without going to all the bother of "naming"; and that when sets are not given to us as collections of well "named" individuals, we can hardly be expected to name representative elements.

But as a result of Zermelo's proof it appears that if one wishes to reject the idea of any well-ordering of the set R of all real numbers, for example, he must, to be consistent, reject the choice axiom. It is now known that numerous other parts of mathematics must simultaneously be rejected in that event. In particular, it is known that the choice axiom, the well-ordering theorem of Zermelo, and the comparability principle* are all equivalent—each implies the other two. So, for example, if we reject the choice axiom, we reject the possibility of comparing in any manner the relative "magnitude" of two arbitrary cardinal numbers.

IV. The continuum hypothesis. Assuming the choice axiom, and hence that every set can be well-ordered, the transfinite cardinal numbers can be considers as "alephs." An *aleph* is the cardinal number of an infinite well-ordered set. For the set N we have already designated the cardinal number by the symbol $\aleph_0$. It is not difficult to show by transfinite induction (see Section 2a above) that if W_1 and W_2 are infinite well-ordered sets with order relations $<_1$, $<_2$ respectively, then one of them, say W_1, has the same ordinal number as part of W_2; and if this part is not all of W_2, we say that $\mathfrak{O}(W_1, <_1) < \mathfrak{O}(W_2, <_2)$.

Denoting the cardinal numbers of W_1, W_2 by $\aleph'$, $\aleph''$ respectively, suppose that by new choices of $<_1$, $<_2$ we have $\mathfrak{O}(W_2, <_2) < \mathfrak{O}(W_1, <_1)$; then we say that $\aleph' = \aleph''$; but if such new choices are impossible, then that $\aleph' < \aleph''$. Then the alephs turn out to be well-ordered with respect to this $<$. The so-called *problem of the continuum* is to determine the position of c as an aleph in this well-ordering.

Let R be well-ordered with respect to an order relation $<_r$. Then if $\mathfrak{O}(N, <)$ is any well-ordering of N, $\mathfrak{O}(N, <) < \mathfrak{O}(R, <_r)$ and it is impossible that $\mathfrak{O}(R, <_r) < \mathfrak{O}(N, <)$ since this would give $c = \aleph_0$.† Hence if c is denoted by $\aleph(c)$, we have $\aleph_0 < \aleph(c)$. The assumption that $\aleph(c) = \aleph_1$ is the usual statement of the continuum hypothesis.

* The comparability principle states that if two totalities are given, there always exists a 1-1-correspondence between one and part of the other.

† It can be shown that the equality as defined in IV is the same as that whose impossibility was proved in III.

CONCERNING SOME PERSPECTIVE TRIANGLES

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1. Introduction. Let $F_1=0$, $F_2=0$, $F_3=0$ be three equations in two variables each, and set

$$L_1 \equiv k_2F_3 - k_3F_2, \qquad L_2 \equiv k_3F_1 - k_1F_3, \qquad L_3 \equiv k_1F_2 - k_2F_1,$$

where k_1, k_2, k_3 are constants. Then $L_1=0$ is satisfied by the simultaneous solutions of $F_2=0$ and $F_3=0$ and also by the simultaneous solutions of $F_1+k_2F_3=0$ and $F_1+k_3F_2=0$. Similar remarks hold for $L_2=0$ and $L_3=0$. Also, since $k_1L_1+k_2L_2+k_3L_3\equiv 0$, it follows that $L_1=0$, $L_2=0$, $L_3=0$ have for simultaneous solutions all simultaneous solutions of any pair of $L_1=0$, $L_2=0$, $L_3=0$.

In each of the following articles we propose to select $F_1=0$, $F_2=0$, $F_3=0$ as equations, in special form, of three positions of some geometric entity, and then to interpret the above algebraic analysis for the resulting figure. This is a useful trick for obtaining a set of different but related theorems. In such a process we are often likely to end up with a projective generalization including all the other cases. At any rate we obtain, in the succeeding articles, some new and rich figures which will call forth some new geometric definitions. We will be able to mention but a few of the theorems suggested.

2. Isogonal lines. Take $F_1=0$, $F_2=0$, $F_3=0$ as cartesian equations, in perpendicular form, of three straight lines, and let the origin of coordinates lie inside triangle $F_1F_2F_3$.

LEMMA. *The lines $F_3+k_1F_2=0$ and $F_2+k_1F_3=0$ are a pair of isogonal lines with respect to angle F_2F_3.*

For let p designate a point on $F_3+k_1F_2=0$ and q a point on $F_2+k_1F_3=0$. Then

$$F_3(p)/F_2(p) = F_2(q)/F_3(q) = -k_1.$$

But, since $F_2=0$ and $F_3=0$ are in perpendicular form, this says that the distances of the points p and q from the lines $F_2=0$ and $F_3=0$ are inversely proportional. This proves the lemma.

THEOREM 1. *If at each vertex of a given triangle ABC a pair of isogonal lines be drawn, then the triangle $A'B'C'$, whose vertices are the points of intersection of pairs of these lines belonging to the same side of the given triangle, is perspective with the given triangle.*

Let $F_1=0$, $F_2=0$, $F_3=0$ be the equations, in perpendicular form, of the three sides of the given triangle. Let the three pairs of isogonal lines be $F_3+k_1F_2=0$ and $F_2+k_1F_3=0$, $F_1+k_2F_3=0$ and $F_3+k_2F_1=0$, $F_2+k_3F_1=0$ and $F_1+k_3F_2=0$. The theorem then follows by article 1.

COROLLARY 1.1. *The Morley triangle of a given triangle is in perspective with the given triangle.*

COROLLARY 1.2. *The altitudes of a triangle are concurrent.*

Reflect each vertex of the given triangle ABC in its opposite side, giving triangle $A'B'C'$. By Theorem 1 triangles ABC and $A'B'C'$ are in perspective.

THEOREM 2. *Given three concurrent cevians, AD, BE, CF, of a triangle ABC. Take any point A' on AD. Let the isogonals of BA' and CA' meet the cevians CF and BE in C' and B' respectively. Then AC' and AB' are isogonal lines for angle A.*

This is an easy converse of Theorem 1.

THEOREM 3. *If m, r, u are concurrent cevians of triangle ABC, then n, s, v, the isogonal lines of m, r, u, are also concurrent cevians of triangle ABC.*

Take A' in Theorem 2 as the intersection of s and u. (These points of concurrency are well known as *isogonal conjugate points* for triangle ABC.)

We state the following three theorems without proof as each is a result easily achieved by long but straightforward analytical geometry. We set $F_1 \equiv a_1x+a_2y-p$, $F_2 \equiv b_1x+b_2y-q$, $F_3 \equiv c_1x+c_2y-r$. (In the present article, of course, we have the restriction $a_1^2+a_2^2=b_1^2+b_2^2=c_1^2+c_2^2=1$.)

THEOREM 4. *If $Q:(\bar{x}, \bar{y})$ is the center of perspectivity, then*

$$\bar{x} = \frac{\begin{vmatrix} p & a_2 & k_1 \\ q & b_2 & k_2 \\ r & c_2 & k_3 \end{vmatrix}}{\begin{vmatrix} a_1 & a_2 & k_1 \\ b_1 & b_2 & k_2 \\ c_1 & c_2 & k_3 \end{vmatrix}}, \qquad \bar{y} = \frac{\begin{vmatrix} a_1 & p & k_1 \\ b_1 & q & k_2 \\ c_1 & r & k_3 \end{vmatrix}}{\begin{vmatrix} a_1 & a_2 & k_1 \\ b_1 & b_2 & k_2 \\ c_1 & c_2 & k_3 \end{vmatrix}}.$$

COROLLARY 4. *The infinitude of triangles determined by the values mk_1, mk_2, mk_3, m a parameter and the k's constants, has a common center of perspectivity with the given triangle.*

THEOREM 5. *The equation of the axis of perspectivity is*

$$\sum [k_1(k_2^2 + k_3^2) + k_2k_3\,(1 + k_1^2)]F_1 = 0,$$

reducing to

$$F_1 + F_2 + F_3 = 0$$

in the case where $k_1=k_2=k_3=k\neq -1, 0$.

COROLLARY 5. *The infinitude of triangles determined by the values $k_1=k_2=k_3=k$, where k is a parameter, has a common center of perspectivity with the given triangle, namely the incenter of the given triangle, and also a common axis of perspectivity with the given triangle.*

THEOREM 6. *The equations of the sides of triangle $A'B'C'$ are*

$$S_1 \equiv k_2k_3(k_1^2 - 1)F_1 + k_1(k_3 + k_1k_2)F_2 + k_1(k_2 + k_3k_1)F_3 = 0,$$

$$S_2 \equiv k_2(k_3 + k_2k_1)F_1 + k_3k_1(k_2^2 - 1)F_2 + k_2(k_1 + k_3k_2)F_3 = 0,$$

$$S_3 \equiv k_3(k_2 + k_1k_3)F_1 + k_3(k_1 + k_2k_3)F_2 + k_1k_2(k_3^2 - 1)F_3 = 0,$$

reducing to

$$S_1 \equiv (k - 1)F_1 + F_2 + F_3 = 0,$$

$$S_2 \equiv F_1 + (k - 1)F_2 + F_3 = 0,$$

$$S_3 \equiv F_1 + F_2 + (k - 1)F_3 = 0,$$

in the case where $k_1 = k_2 = k_3 = k \neq -1, 0$.

COROLLARY 6. *For* $k_1 = k_2 = k_3 = 2$ *we have* A', B', C' *collinear.*

The figure for this corollary is very interesting. The three pairs of isogonals form a hexagon about ABC, and the three pairs of opposite sides of this hexagon meet in three collinear points, lying on the axis of perspectivity of the given triangle and the triangle formed by the feet of its angle bisectors.

3. Isotomic points. Take $F_1 \equiv a_1u + a_2v + 1 = 0$, $F_2 \equiv b_1u + b_2v + 1 = 0$, $F_3 \equiv c_1u + c_2v + 1 = 0$, Plücker line equations, in non-homogeneous form, of three points A, B, C.

LEMMA. *The points* $F_3 + k_1F_2 = 0$ *and* $F_2 + k_1F_3 = 0$ *are a pair of isotomic points with respect to the segment* F_2F_3.

For the cartesian coordinates of points $F_2 = 0$ and $F_3 = 0$ are (b_1, b_2) and (c_1, c_2) respectively. Similarly, the cartesian coordinates of the points $F_3 + k_1F_2 = 0$ and $F_2 + k_1F_3 = 0$ are

$$\left(\frac{c_1 + k_1b_1}{1 + k_1}, \frac{c_2 + k_1b_2}{1 + k_1}\right) \text{ and } \left(\frac{b_1 + k_1c_1}{1 + k_1}, \frac{b_2 + k_1c_2}{1 + k_1}\right)$$

respectively. These points are obviously isotomic points for the segment F_2F_3.

THEOREM 1. *If on each side of a given triangle ABC a pair of isotomic points be taken, then the triangle $A'B'C'$, whose sides are the joins of these points belonging to the same vertex of the given triangle, is perspective with the given triangle.*

This is merely an interpretation of Article 1 where $F_1 = 0$, $F_2 = 0$, $F_3 = 0$ are the Plücker line equations, in non-homogeneous form, of the vertices of the given triangle. This theorem is a sort of dual of Theorem 1 Article 2.

As the corresponding duals of Theorems 2 and 3 Article 2 we have the following result.

THEOREM 2. *Let a line cut the sides BC, CA, AB of a triangle ABC in the points D, E, F respectively. Through F draw a line cutting AC and BC in R and M respectively. Take N on BC isotomic with respect to M, and S on AC isotomic with respect to R. Let EN and DS cut AB in U and V respectively. Then U and V are isotomic points on AB.*

THEOREM 3. *If M, R, U are collinear points on the sides of a triangle ABC, then N, S, V, the isotomic points of M, R, U, are also collinear points on the sides of triangle ABC.*

DEFINITION. We will call the two lines of collinearity of Theorem 3 *isotomic conjugate lines* of triangle *ABC.*

From Theorems 4 and 5 Article 2 we now obtain the following theorem.

THEOREM 4. *If $q:(\bar{u}, \bar{v})$ is the axis of perspectivity, then*

$$\bar{u} = -\frac{\begin{vmatrix} 1 & a_2 & k_1 \\ 1 & b_2 & k_2 \\ 1 & c_2 & k_3 \end{vmatrix}}{\begin{vmatrix} a_1 & a_2 & k_1 \\ b_1 & b_2 & k_2 \\ c_1 & c_2 & k_3 \end{vmatrix}}, \qquad \bar{v} = -\frac{\begin{vmatrix} a_1 & 1 & k_1 \\ b_1 & 1 & k_2 \\ c_1 & 1 & k_3 \end{vmatrix}}{\begin{vmatrix} a_1 & a_2 & k_1 \\ b_1 & b_2 & k_2 \\ c_1 & c_2 & k_3 \end{vmatrix}}.$$

COROLLARY 4. *The infinitude of triangles determined by the values mk_1, mk_2, mk_3, m a parameter and the k's constants, has a common axis of perspectivity with the given triangle.*

THEOREM 5. *The Plücker equation, in non-homogeneous form, of the center of perspectivity is*

$$\frac{\sum [k_1(k_2^2 + k_3^2) + k_2k_3(1 + k_1^2)]F_1}{\sum [k_1(k_2^2 + k_3^2) + k_2k_3(1 + k_1^2)]} = 0,$$

reducing to

$$(F_1 + F_2 + F_3)/3 = 0$$

in the case where $k_1 = k_2 = k_3 = k \neq -1, 0$.

COROLLARY 5. *The infinitude of triangles determined by the values $k_1 = k_2 = k_3 = k$, say, where k is a parameter, has a common center of perspectivity with the given triangle, namely the common centroid of all the triangles and of the given triangle, and also a common axis of perspectivity with the given triangle, namely the line at infinity.*

We note that Corollary 6 Article 2 becomes trivial here.

4. Isotomic lines. Take $F_1 \equiv y-ax-p=0$, $F_2 \equiv y-bx-q=0$, $F_3 \equiv y-cx-r=0$, cartesian equations, in slope-intercept form, of three straight lines.

DEFINITION. Given an angle BAC and a line L not through vertex A, cutting AB and AC in M and N respectively. Let U and V be a pair of isotomic points with respect to the segment MN. Then the lines AU and AV will be known as a pair of *isotomic lines* of the angle BAC with respect to the line L.

(The concept of isotomic lines is one worth introducing into College Geometry. Note that isotomic lines of angle BAC with respect to line L are also isotomic lines of angle BAC with respect to any line parallel to L and not passing through A. Also note that isogonal lines of angle BAC are isotomic lines of angle BAC with respect to any line not passing through A and perpendicular to the angle bisector of angle BAC. Any two lines through the vertex A are isotomic lines of angle BAC for some line L.)

LEMMA. *The lines $F_3+k_1F_2=0$ and $F_2+k_1F_3=0$ are a pair of isotomic lines of the angle F_2F_3 with respect to the y-axis.*

For we have

$$F_3 + k_1F_2 \equiv y\,(1 + k_1) - (c + k_1b)x - (r + k_1q) = 0.$$

Hence the y-intercept of $F_3+k_1F_2=0$ is $(r+k_1q)/(1+k_1)$. Similarly, the y-intercept of $F_2+k_1F_3=0$ is $(q+k_1r)/(1+k_1)$. This proves the lemma.

THEOREM 1. *Given a triangle ABC and a line L not through a vertex, and a pair of isotomic lines through each vertex with respect to the line L. Then the triangle $A'B'C'$, whose vertices are the points of intersection of pairs of these lines belonging to the same side of the given triangle, is perspective with the given triangle.*

This is merely an interpretation of Article 1 where $F_1=0$, $F_2=0$, $F_3=0$ are the cartesian equations, in slope-intercept form, of the sides of the given triangle, and where L is selected as the y-axis.

Theorems 4 and 5 of Article 2 now yield the following theorem.

THEOREM 2. *If $Q:(\bar{x}, \bar{y})$ is the center of perspectivity, then*

$$\bar{x} = \frac{\begin{vmatrix} p & 1 & k_1 \\ q & 1 & k_2 \\ r & 1 & k_3 \end{vmatrix}}{\begin{vmatrix} -a & 1 & k_1 \\ -b & 1 & k_2 \\ -c & 1 & k_3 \end{vmatrix}}, \qquad \bar{y} = \frac{\begin{vmatrix} -a & p & k_1 \\ -b & q & k_2 \\ -c & r & k_3 \end{vmatrix}}{\begin{vmatrix} -a & 1 & k_1 \\ -b & 1 & k_2 \\ -c & 1 & k_3 \end{vmatrix}}.$$

COROLLARY 2.1. *If $k_1:k_2:k_3=p:q:r$, then Q is at the origin; that is, the center of perspectivity lies on L.*

COROLLARY 2.2. *If $k_1=k_2=k_3=k$, then Q is at infinity in the direction of L.*

COROLLARY 2.3. *The infinitude of triangles determined by the values mk_1, mk_2, mk_3, where m is a parameter and the k's are constants, has a common center of perspectivity with the given triangle.*

THEOREM 3. *The cartesian equation, in slope-intercept form, of the axis of perspectivity is given by the equations of Theorem 5 Article 3.*

COROLLARY 3. *If $k_1=k_2=k_3=k$, then the y-intercept of the axis of perspectivity is $(p+q+r)/3$. That is, the axis of perspectivity cuts L in the centroid of the points where the sides of the triangle cut L.*

THEOREM 4. *Given a triangle ABC and a line L cutting the sides BC, CA, AB in D, E, F respectively. Take points V and W on L such that $FV/VE=FW/WD$. Let AV and BW meet in C'. Then CC' is parallel to L.*

By Corollary 2.2 we have the next theorem.

THEOREM 5. *If m, r, u are concurrent cevians of triangle ABC, then n, s, v, the isotomic lines of m, r, u for a given line L, are also concurrent cevians of triangle ABC.*

DEFINITION. We will call the two points of concurrency of Theorem 5 a pair of *isotomic conjugate points* of triangle *ABC for the line L.*

THEOREM 6. *If $k_1=k_2=k_3=2$, then A', B', C' are collinear.*

The figure for this theorem is very interesting.

5. Isogonal points. Without pausing to set up the analytical geometry we merely state that isotomic lines may be dualized by the following definition.

DEFINITION. Given a line segment BC and a point P not on BC. Select on BC two points M and N such that PM and PN are isogonal lines for angle BPC. Then M and N will be known as a pair of *isogonal points* of the segment BC with respect to the point P. (This is another concept worth introducing into College Geometry.)

The dual of Theorem 1, Article 4 then appears as follows.

THEOREM 1. *Given a triangle ABC and a point P not on any side, and a pair of isogonal points with respect to P on each side of the triangle. Then the triangle $A'B'C'$, whose sides are the joins of these points belonging to the same vertex of the given triangle, is perspective with the given triangle.*

6. Isoanharmonic lines. Let $F_1\equiv a_1x+a_2y-1=0$, $F_2\equiv b_1x+b_2y-1=0$, $F_3\equiv c_1x+c_2y-1=0$, cartesian equations, in intercept form, of three straight lines.

DEFINITION. Given an angle BAC and a point P not on the sides of the angle. Let AM and AN be lines such that $A(PBMC)=A(PCNB)$. Then the lines AM and AN will be known as a pair of *isoanharmonic lines* of angle A with respect to point P.

LEMMA. *The lines* $F_3+k_1F_2=0$, $F_2+k_1F_3=0$ *are a pair of isoanharmonic lines of the angle* F_2F_3 *with respect to the origin.*

For we have

$$F_3 + k_1F_2 \equiv (c_1 + k_1b_1)x + (c_2 + k_1b_2)y - (1 + k_1) = 0.$$

Hence the x-intercept of $F_3+k_1F_2=0$ is $(1+k_1)/(c_1+k_1b_1)$. Similarly, the x-intercept of $F_2+k_1F_3=0$ is $(1+k_1)/(b_1+k_1c_1)$. Then we have

$$\left(0, \frac{1}{c_1}, \frac{1+k_1}{c_1+k_1b_1}, \frac{1}{b_1}\right) = \frac{1}{k_1},$$

$$\left(0, \frac{1}{b_1}, \frac{1+k_1}{b_1+k_1c_1}, \frac{1}{c_1}\right) = \frac{1}{k_1}.$$

Hence the lemma is proved.

THEOREM 1. *Given a triangle* ABC *and a point* P *not on any of the sides, and through each vertex a pair of isoanharmonic lines with respect to the point* P. *Then the triangle* $A'B'C'$, *whose vertices are the points of intersection of pairs of these lines belonging to the same side of the given triangle, is perspective with the given triangle.*

This is merely an interpretation of Article 1 where $F_1=0$, $F_2=0$, $F_3=0$ are the cartesian equations, in intercept form, of the sides of the given triangle, P being the origin of coordinates.

THEOREM 2. $Q(\bar{x}, \bar{y})$ *is the center of perspectivity, then*

$$\bar{x} = \frac{\begin{vmatrix} 1 & a_2 & k_1 \\ 1 & b_2 & k_2 \\ 1 & c_2 & k_3 \end{vmatrix}}{\begin{vmatrix} a_1 & a_2 & k_1 \\ b_1 & b_2 & k_2 \\ c_1 & c_2 & k_3 \end{vmatrix}}, \qquad \bar{y} = \frac{\begin{vmatrix} a_1 & 1 & k_1 \\ b_1 & 1 & k_2 \\ c_1 & 1 & k_3 \end{vmatrix}}{\begin{vmatrix} a_1 & a_2 & k_1 \\ b_1 & b_2 & k_2 \\ c_1 & c_2 & k_3 \end{vmatrix}}.$$

COROLLARY 2. *If we take* $k_1=k_2=k_3=k$, *then* Q *is at the origin; that is,* Q *coincides with* P.

THEOREM 3. *Article 6 is a projective generalization of Articles 2 and 4. In Article 2,* P *is at the incenter of triangle* ABC; *in article 4,* P *is at infinity in the direction of line* L.

7. Isoanharmonic points. We now dualize isoanharmonic lines by the

DEFINITION. Given a line segment BC and a line L not through B or C. Let M and N be points on BC such that $(LBMC)=(LCNB)$. Then the points M and N will be known as a pair of *isoanharmonic points* of the segment BC with respect to the line L.

Our principal theorem becomes the following.

THEOREM 1. *Given a triangle ABC and a line L not through any of its vertices, and on each side a pair of isoanharmonic points with respect to the line L. Then the triangle $A'B'C'$, whose sides are the joins of these points belonging to the same vertex of the given triangle, is perspective with the given triangle.*

THEOREM 2. *Article 7 is a projective generalization of articles 3 and 5.*

8. Isopotential circles. Take $F_1=0$, $F_2=0$, $F_3=0$ as cartesian equations, in standard form, of three circles.

DEFINITION. Given two circles whose equations, in standard form, are $F_2=0$ and $F_3=0$. Then the two circles $S_{23}\equiv F_3+k_1F_2=0$, $S_{32}\equiv F_2+k_1F_3=0$ (which are coaxal with the given circles) will be known as a pair of *isopotential circles.*

We note that circle S_{23} is the locus of points the ratio of whose powers with respect to circles F_2 and F_3 is $-k_1$, and S_{32} is the locus of points the ratio of whose powers with respect to circles F_3 and F_2 is also $-k_1$. This geometrically justifies the name *isopotential circles.*

Our principal theorem, now somewhat complex, assumes the following form.

THEOREM. *Given three circles F_1, F_2, F_3 and the circles*

$$S_{23}, S_{32}, \text{ isopotentials of } F_2 \text{ and } F_3,$$

$$S_{31}, S_{13}, \text{ isopotentials of } F_3 \text{ and } F_1,$$

$$S_{12}, S_{21}, \text{ isopotentials of } F_1 \text{ and } F_2.$$

Then there exist circles L_1, L_2, L_3 such that

$$L_1 \text{ is coaxal with } F_2, F_3 \text{ and } S_{12}, S_{13},$$

$$L_2 \text{ is coaxal with } F_3, F_1 \text{ and } S_{23}, S_{21},$$

$$L_3 \text{ is coaxal with } F_1, F_2 \text{ and } S_{31}, S_{32},$$

$$L_1, L_2, L_3 \text{ are themselves coxal.}$$

COROLLARY. *If $k_1=k_2=k_3=-1$ we have the familiar situation of the three radical axes of three circles taken in pairs being concurrent.*

RATIONAL POINTS ON CUBIC CURVES AND SURFACES*

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1. Introduction. Let $f \equiv f(x_1, x_2, \cdots, x_n)$ be a cubic polynomial with rational coefficients in the n variables $x_1, x_2, \cdots, x_n$, where $n = 2$ or 3, and where we suppose that f is irreducible and is not a function of less than n variables. We deal with the rational solutions of the equation $f = 0$, *i.e.*, the rational points on the manifold $f = 0$. On writing $x_1 = X_1/X_{n+1}, \cdots, x_n = X_n/X_{n+1}$, we may suppose that the X's are integers with $X_{n+1} \neq 0$, and then the equation $f = 0$ becomes a homogeneous cubic equation

$$g \equiv g(X_1, X_2, \cdots, X_{n+1}) = 0$$

in $n+1$ variables. The problems of finding the integral solutions of $g = 0$ with $X_{n+1} \neq 0$, and the rational ones of $f = 0$ are identical.

2. Plane cubics. Suppose first that $n = 2$. Write the equation as $f \equiv f(x, y) = 0$, thus defining a plane irreducible cubic curve. The curve has either one double point, (a, b) say, or none. In the first case all the rational points can be found very simply. It is easy to show that a, b are rational numbers. Then there is obviously a one-one correspondence between all the rational points on the curve and the lines $y - b = m(x - a)$, where m is any rational number. The intersection of this line with the curve gives x, y as rational functions of m with rational coefficients.

When the curve has no double point, no general criterion is known for the solvability of $f = 0$, nor is any method known for finding even one solution, even though the equation has attracted the attention of mathematicians for at least three centuries and in some form or other has been the subject of innumerable papers.

When, however, a set of one or more rational points $P_1, P_2, P_3, \cdots$ is known on $f = 0$, *e.g.*, by inspection, in general an infinity of other rational solutions can be found by a process known in principle for many centuries. Thus unless P_1 is a point of inflexion, the tangent at P_1 meets the curve in just one other point different from P_1, say $P_{1,1}$, and this is a rational point since it is determined by a rational equation of the first degree. Again, from two points P_1, P_2, neither of which lies on the tangent at the other, a third rational point is given by the remaining intersection, say $P_{1,2}$, of the chord P_{12} and the cubic. The chord and tangent process can be applied not only to those of the original set but also to all the points found in this way. In general, an infinity of rational points will be found though in exceptional cases the points so found form a finite polygon.

The process can also be described by simple algebra without any mention of geometry and was so known to Diophantos about 250 A. D. Suppose, *e.g.*, rational points $(p, \pm q)$, $q \neq 0$, are known on the curve

$$y^2 = ax^3 + bx^2 + cx + d.$$

* Lecture given to the Archimedeans, Cambridge University, England, October 20, 1943.

On transforming the origin to $(p, 0)$, we may suppose that $d=q^2$. We now seek for the particular solution with $y=q+(c/2q)x$. On substituting and dividing by x^2, we have a linear equation for x. Similarly, on assuming that

$$y = q + \frac{c}{2q}x + \lambda x^2,$$

where

$$2\lambda q + \frac{c^2}{4q^2} = b,$$

substituting and dividing by x^3, we have a linear equation in x. It is not difficult to derive this value from that found above on applying to it and its reflection in the x axis the chord and tangent process.

The known rational point may be at infinity. Thus for the cubic curve

$$y^3 = a^3x^3 + bx^2 + cx + d,$$

where b, c are not both zero, the assumption $y=ax+b/3a^2$ leads to a linear equation for the finite point of intersection of the curve and its asymptote $y=ax+b/3a^2$ when the point at infinity is not an inflexion.

3. The finite basis theorem. In 1901, Poincaré made the conjecture that all the rational points on a cubic curve without a double point could be deduced from a finite number by repeated application of the chord and tangent process. Probably this finite basis conjecture had been known long before. I proved it 22 years ago and give now a brief outline of the proof. It depends on the consideration of the rational points on the quartic curve

$$y^2 = ax^4 + bx^3 + cx^2 + dx + e.$$

Here, as for the cubic, on putting $e=q^2$, the assumption

$$y = q + \frac{d}{2q}x + \left(c - \frac{d^2}{4q^2}\right)\Big/ 2q$$

in general leads to a new rational point.

Suppose now that (p, q) is a rational solution of the cubic equation $f(x, y)=0$. Put $x=p+tu$, $y=q+t$. Then

$$t^2S_3 + tS_2 + S_1 = 0,$$

where S_1, S_2, S_3 are polynomials of degree 1, 2, 3 in u. Hence

$$2tS_3 + S_2 = v,$$

where

$$v^2 = S_2^2 - 4S_1S_3.$$

This is an equation of the form

$$v^2 = au^4 + bu^3 + cu^2 + du + e.$$

On putting $u=X/Y$, $v=Z/Y^2$, where X, Y, Z are integers without a common factor, and $a=A/F, \cdots, e=E/F$, where $A, \cdots, F$ are integers, it takes the form

$$FZ^2 = AX^4 + BX^3Y + CX^2Y^2 + DXY^3 + EY^4.$$

Let θ be a root of the equation

$$A\theta^4 + B\theta^3 + C\theta^2 + D\theta + E = 0.$$

Then

$$(X - \theta Y)(AX^3 + X^2Y(B + A\theta) + \cdots) = FZ^2.$$

Now from an equation of the form $UV=FW^2$, where U, V, W are integers and $(U, V)=1$, we find easily

$$U = \pm F_1P^2, \qquad V = \pm F_2Q^2,$$

where P, Q are any integers and F_1, F_2 belong to a finite set of integers. But the procedure is not so simple when the factors U, V are algebraic numbers. Though it is easy to define algebraic integers, their factorization leads to new principles and requires a deep study of the arithmetic theory of algebraic numbers. Then we can deduce an equation of the form

$$m(X - \theta Y) = (\alpha_0 + \beta_0\theta + \gamma_0\theta^2 + \delta_0\theta^3)(\alpha + \beta\theta + \gamma\theta^2 + \delta\theta^3)^2,$$

where α, β, γ, δ are any integers and α_0, β_0, γ_0, δ_0, m belong to a finite set of integers. Then if the original equation has an infinity of integral solutions, and if the set α_0, β_0, γ_0, δ_0, m occurs also for the particular solution X_0, Y_0, say, we deduce

$$n^2(X - \theta Y)(X_0 - \theta Y_0) = (A + B\theta + C\theta^2 + D\theta^3)^2,$$

where A, B, C, D are variable integers, and n is fixed.

On multiplying out and replacing θ^4, θ^5, θ^6 in terms of 1, θ, θ^2, θ^3 and equating terms on both sides, we deduce two equations between the unknowns A, B, C, D. Then it is shown that certain linear functions of A, B, C, D lead to new solutions x_1, y_1 of the equation, and we have

$$X = R_1(X_1, Y_1, Z_1), \qquad Y = R_2(X_1, Y_1, Z_1), \qquad Z = R_3(X_1, Y_1, Z_1),$$

where the R's are rational functions with rational coefficients. From these we can deduce

$$\max(|X_1|, |Y_1|) < k(\max(|X|, |Y|))^{1/2}$$

where k depends on X_0, Y_0. By continuing this process, called the method of infinite descent, with the solution X_1, Y_1, Z_1, we are led to a solution X_n, Y_n with $\max.(|X_n|, |Y_n|) \leqq k^2$, and there can be only a finite number of such solutions. Further, the relations between the solutions X, Y, Z; X_1, Y_1, Z_1, *etc.*, can be proved to be essentially in accordance with the tangent and chord process.

The finite basis theorem can be put in a very simple analytical form if the cubic be reduced, as it may when one solution is known, to that of the canonical form

$$y^2 = 4x^3 - g_2x - g_3.$$

Then $x = \wp(u)$, $y = \wp'(u)$ is the usual parametric representation by elliptic functions, and all the rational solutions are given by taking $u = n_1u_1 + n_2u_2 + \cdots + n_ru_r$, where $u_1, u_2, \cdots, u_r$ are a finite number of constants and $n_1, n_2, \cdots, n_r$ run through all integer values.

4. Cubic surfaces. We turn now to the cubic surface, say $f(x, y, z) = 0$, or $g(x, y, z, w) = 0$ in the homogeneous form. We exclude the cases when $f = 0$ is a cone, *i.e.*, when the surface has a triple point; and when $f = 0$ is a cylinder, *i.e.*, f is essentially a function of only two variables, since the problems then become those of the plane cubic. There are some simple cases when all the rational points are easily found, *e.g.*, when the surface has a rational double point. About 1756, Euler found a solution of the equation

$$x^3 + y^3 + z^3 + w^3 = 0,$$

a subject included in elementary books on number theory. As shown substantially by Hermite, a solution can be easily found for any cubic surface if, as in Euler's case, there exist on it two non-intersecting lines, the coefficients of whose equations are either rational numbers or conjugate quadratic numbers. For a one-one correspondence can be established between all the rational points P of the surface other than those on the two lines, and the rational points Q on any plane with rational coefficients, by imposing the conditions that the line PQ meets each of the lines.

Next, Kraft, Lagrange and Euler, about 1770, solved equations of the form

$$\prod_{\theta,\phi,\psi} (x + \theta y + \theta^2 z) = w^3,$$

where θ, ϕ, ψ are roots of a cubic equation with rational coefficients. A parametric solution in terms of three rational numbers A, B, C is given by taking

$$x + \theta y + \theta^2 z = (A + B\theta + C\theta^2)^3,$$
$$w = \prod (A + B\theta + C\theta^2).$$

Suppose next that a rational point P is known on the surface $f = 0$. The tangent plane at P meets the surface in general in an irreducible plane cubic curve with a double point at P and so we can at once find an infinity of rational points depending on a rational parameter. This result was given essentially by Libri in 1820, but was probably known before that time.

In 1826, Ryley found a parametric solution of the equation $x^3 + y^3 + z^3 = n$ by the aid of a most ingenious artifice whose significance seemed very obscure until fairly recently.

5. Recent developments. After Ryley's work, the subject lay dormant for many years. Then in 1930 another solution of Ryley's equations was given by Richmond who sought solutions in which x, y, z were proportional to cubic polynomials in a parameter t, and Ryley's solution was found to fit in with those so found. The success of Richmond's method depended on the existence of three rational solutions of $x^3+y^3+z^3=0$, namely the inflexion $(1, -1, 0)$, *etc.*

The last year has been fruitful in new advances by B. Segre and myself, accounts of which are appearing in the Journal of the London Mathematical Society. For the first time since the 120 years that elapsed since Ryley's work, a new type of equation was solved parametrically. This was

$$(x + y + z)^3 - dxyz = m, \qquad (d \neq 0)$$

which includes Ryley's as a special case as is clear on putting

$$x = Y + Z, \quad y = Z + X, \quad z = X + Y, \quad d = 24, \quad m = 8r.$$

I found a particular solution on taking $m=dyz^2$. Then

$$(x + y + z)^3 = dyz(x + z),$$

which in homogeneous coordinates is a cubic curve with a double point at $(-1, 0, 1)$. The curve meets the general line $x+z=py$ through the double point, where $(p+1)^3y^3=pdy^2z$. Hence

$$\frac{y}{z} = \frac{dp}{(p + 1)^3},$$

and so

$$\frac{x}{z} = \frac{dp^2}{(p + 1)^3} - 1.$$

Then from $m=dyz^2$,

$$\frac{m}{z^3} = \frac{d^2p}{(p + 1)^3}.$$

Since z is to be rational, $p=dmt^3$, where t is a new parameter. Hence

$$z = \frac{dmt^3 + 1}{dt}, \quad y = \frac{dmt^2}{(dmt^3 + 1)^2}, \quad x = \frac{d^2m^2t^5}{(dmt^3 + 1)^2} - \frac{dmt^3 + 1}{dt}$$

is a parametric solution.

Then B. Segre found a parametric solution of the equation

$$x^3 + y^3 + az^3 = b, \qquad (b \neq 0)$$

by noting that a solution could be deduced from one for

$$xy(x + y) + az^3 = b$$

by a well known transformation connecting the forms

$$x^3 + y^3, \qquad XY(X + Y).$$

It is now possible to apply the geometric ideas of Richmond's paper, since the curve $xy(x+y)+az^3=0$ in homogeneous coordinates has three rational points of inflexion given by $z=0$, $xy(x+y)=0$.

Then Segre made a systematic study of the arithmetical properties of cubic surfaces, and found many new and important results of which I mention here only a few. Starting from Libri's result, he proved that every cubic surface contained either 0, 1, 3 or an infinity of rational points. Surfaces with no rational points exist (and this also holds for eight dimensional manifolds as shown by Hasse and also by myself), and a particular case,

$$x^3 + 2y^3 = 7(z^3 + 2w^3),$$

of this result is given by Segre. We may suppose that $(x, y, z, w)=1$. Then since to mod 7,

$$x^3 \equiv 0, \pm 1, \qquad y^3 \equiv 0, \pm 1, \qquad x^3 + 2y^3 \equiv 0,$$

clearly $x \equiv y \equiv 0$. Then since $z^3+2w^3 \equiv 0$, $z \equiv w \equiv 0$. Hence $(x, y, z, w) \equiv 0$, a contradiction.

He showed next that surfaces with only 3 rational points, if any exist, could be reduced to a canonical form, substantially,

$$z^2 - (1 + px)^2 = ax^3 + bx^2y + cxy^2 + dy^3,$$

the three rational points being $(0, 0, \pm 1)$ and the point at infinity where their join meets the surface. He then established the existence of a parametric solution with 3 parameters and so every surface has 0, 1 or an infinity of rational solutions. He did this by considering the 18 intersections of the surface with suitably chosen quadric and cubic surfaces, showing that 17 of these intersections can be rationally assigned and so the eighteenth can be found by rational elimination processes.

Immediately afterwards, when I knew that the equation could be solved, I found a rational parametric solution by simple algebra, and then applied the same idea to the equation

$$z^2 - k(1 + px)^2 = ax^3 + bx^2y + cxy^2 + dy^3.$$

This is a particular case of the equation

$$z^2 = f(x, y),$$

where f is a cubic polynomial in x, y with rational coefficients. Segre showed that every cubic surface with only one rational point could be reduced to this form. He then applied my method to deduce a rational parametric solution of the equation and so finally proved that every cubic surface has either none or an infinity of rational solutions.

6. An illustration. I illustrate my method by finding a rational parametric solution of the equation

$$z^2 - k = ax^3 + by^3.$$

We may suppose that $ab \neq 0$ as otherwise the question reduces to that of the cubic curve, and also that $k \neq 0$ since otherwise a rational solution is obvious on putting $x = pz$, $y = qz$. On putting $x = X/a$, $z = Z/a$, we may take $a = 1$.

Let θ, ϕ, ψ be the roots of $\zeta^3 - b = 0$. Then

$$(z - \sqrt{k})(z + \sqrt{k}) = \prod_{\theta,\phi,\psi} (x + \theta y).$$

We can satisfy this by putting

$$z + \sqrt{k} = \prod_{\theta,\phi,\psi} (x_1 + \theta y_1 + \theta^2 z_1 + \sqrt{k}(x_2 + \theta y_2 + \theta^2 z_2)),$$

$$z - \sqrt{k} = \prod_{\theta,\phi,\psi} (x_1 + \theta y_1 + \theta^2 z_1 - \sqrt{k}(x_2 + \theta y_2 + \theta^2 z_2)),$$

where $x_1, y_1, z_1, x_2, y_2, z_2$ are rational numbers, and

$$x + \theta y = (x_1 + \theta y_1 + \theta^2 z_1)^2 - k(x_2 + \theta y_2 + \theta^2 z_2)^2.$$

I may remark that every cubic equation with only a finite positive number of solutions can be reduced to the form $z^2 = f(x, y)$. The existence of any solution of the homogeneous cubic leads at once by an obvious rational linear transformation to an equation of the form

$$z^2 L_1 + z L_2 + L_3 = 0,$$

where L_1, L_2, L_3 are homogeneous polynomials of the first, second, third degrees respectively in x, y, w. Now L_1, L_2 are not both identically zero since it has been already assumed that the surface is not a cylinder. Next L_1 is not identically zero since then there would be an infinity of integral solutions given by taking arbitrary x, y, w for which $L_2 \neq 0$, and so by an appropriate linear transformation we can write $L_1 = w$. Next we have an infinity of solutions on taking $w = 0$ unless this implies $L_2 = 0$, *i.e.*, L_2 is divisible by w. But putting $z - L_2/w$ for z, the cubic equation takes the required form

$$\left(\frac{z}{w}\right)^2 = f\left(\frac{x}{w}, \frac{y}{w}\right),$$

where f is a cubic polynomial in x/w, y/w. On putting in here $\theta^3 = b$, $\theta^4 = \theta b$, and equating the coefficient of θ^2 to zero, we obtain

$$y_1^2 + 2x_1 z_1 = k(y_2^2 + 2x_2 z_2).$$

Also from the equation for $z \pm \sqrt{k}$, by subtracting, we get

$$1 = k \prod_{\theta,\phi,\psi} (x_2 + \theta y_2 + \theta^2 z_2) + \sum_{\theta,\phi,\psi} (x_2 + \theta y_2 + \theta^2 z_2)(x_1 + \phi y_1 + \phi^2 z_1)(x_1 + \psi y_1 + \psi^2 z_1).$$

This becomes on multiplying out

$$1 = k(x_2^3 + by_2^3 + b^2z_2^3 - 3bx_2y_2z_2) + x_2(x_1^2 - 3by_1z_1) + y_2(3by_1^2 - 3bx_1z_1) + z_2(3b^2z_1^2 - 3bx_1y_1).$$

Obviously any rational solution of these two equations in the six variables $x_1, y_1, z_1, x_2, y_2, z_2$ gives rational values of x, y, z. For a particular one take $x_2=0$, whence $x_1=(ky_2^2-y_1^2)/2z_1$. Substituting in the other equation, we have

$$1 = kby_2^3 + kb^2z_2^3 + 3by_1^2y_2 - \frac{3b}{2}y_2(ky_2^2 - y_1^2) + 3b^2z_1^2z_2 - \frac{3by_1z_2}{2z_1}(ky_2^2 - y_1^2).$$

Put $y_2=1/Y_2$, $y_1=Y_1/Y_2$, $z_1=Z_1/Y_2$, $z_2=Z_2/Y_2$. Then

$$Y_2^3 = kb + kb^2Z_2^3 + 3bY_1^2 + 3b^2Z_1^2Z_2 - \frac{3b}{2}k + \frac{3b}{2}Y_1^3 - \frac{3b}{2}\frac{Y_1}{Z_1}Z_2(k - Y_1^2).$$

On putting $3bZ_2/2Z_1=\lambda^3$ and taking Z_1, λ to be parameters, we have the equation in Y_1, Y_2

$$Y_2^3 = \lambda^3Y_1^3 + \frac{9b}{2}Y_1^2 - k\lambda^3Y_1 + kb^2Z_2^3 + 3b^2Z_1^3Z_2 - \frac{kb}{2},$$

of which a particular solution is given on taking

$$Y_2 = \lambda Y_1 + \frac{3b}{2\lambda^2}.$$

Hence we have found a new parametric solution of the equation.

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Compact Tables. Arranged and published by J. P. Ballantine. Seattle, University of Washington, 1943. 38 pages. $0.25.

Compact Tables. Second Edition. By J. P. Ballantine. Chicago, The Charles T. Powner Company, 1943. 46 pages. $0.50.

The first edition contains the following five-place tables: logarithms of numbers, natural trigonometric functions, logarithms of trigonometric functions. The wide margins at the bottoms of the pages are filled with a variety of odds and ends: values and logarithms of π and e and their reciprocals, a few weights and measures, formulas from plane and spherical trigonometry, plane analytical geometry, and differential and integral calculus. Entries are in large sharp print on serviceable paper. The booklet is equipped with tabs for ready reference (these, however, are not reinforced) and is bound in a stiff paper cover. The logarithms of the trigonometric functions are given by separate functions, 10° to the page; for example, page 26 contains log sines for angles between 20° and 30° and log cosines for angles between 60° and 70°. One noticeable feature of these logarithmic trigonometric tables is the absence of proportional parts; only differences are given. Instead, a nomogram, to be found at the back of the booklet, may be used (with the help of a straight-edge) for interpolation in seconds or decimal parts of a degree. It should also be noted that the table of logarithmic tangents goes only as far as 45°.

In the second edition the table of logarithmic tangents is extended to 90°, and two tables are added: namely, a five-place table of natural logarithms and a six-place table giving the conversion from degrees, minutes, and seconds to radians.

The reviewer made no attempt to check the entries.

R. H. BRUCK

Method-pamphlets on the Milne Method of Numerical Integration of First-order Differential Equations and of Certain Equations of Second Order. Oakland, California, Marchant Calculating Machine Company, 1944. 4 pamphlets: MM-216, MM-216A, MM-260, MM-261. 10, 6, 11, 4 pages. No charge.

The topics dealt with in these pamphlets are as follows:

MM-216, Integration of $dy/dx=f(x, y)$ when starting values are known;

MM-216A, Integration of $d^2y/dx^2=g(x, y)$ when starting values are known;

MM-260 and MM-261, Obtaining starting values for use in MM-216 and MM-216A.

These pamphlets outline methods by W. E. Milne for numerical integration of ordinary equations in a form which may be useful to computors to whom calculating machines are available. The material includes sample work sheets and statements about the reliability of the results in terms of differences for the starting values. The pamphlets contain references to National Council Research Bulletin, No. 92, 1933, by A. A. Bennett, W. E. Milne, and H. Bateman and to papers by W. E. Milne in this MONTHLY 1933, 1941, 1942.

L. W. COHEN

"Student's" Collected Papers. Edited by E. S. Pearson and John Wishart, with a Foreword by Launce McMullen. London, Biometrika Office, University College, 1943. 14+224 pages. 15s.

This book is well described by the following summary which appears on the jacket:

"The death in 1937 of William Sealey Gosset dealt a final blow to the anonymity which had concealed the personality of 'Student,' the statistician. Under this pseudonym his work had become famous wherever biological experiments were undertaken and the modern theory of statistics, with its emphasis on the valid deductions that can be made from the 'small sample,' was studied. 'Student's' test and 'Student's' tables have become household words in this field. They are included in their original form in this book, which is a re-issue, with Foreword, of all the scientific papers published by him over a period of 30 years. A unity of purpose runs through the whole of his contributions, and the result is, in retrospect, a valuable addition to the structure of the Science of Statistics."

It was his second paper (1908), *On The Probable Error of a Mean*, that made "Student" famous. Apart from its own merits as a notable piece of research, it played an important role in stimulating further contributions. This paper may be said to mark the beginning of the modern theory dealing with small samples and tests of statistical hypotheses. In all, "Student" published 21 papers and a few miscellaneous contributions. Although 14 of these papers appeared in *Biometrika*, the remainder were scattered throughout several other journals. We are thus indebted to the editors for making possible a balanced appraisal of the work of a versatile pioneer by collecting these papers into one volume. The central theme of his contributions was the application of statistical method to the research and routine problems of industry and agriculture. It would be surprising if the work of a pioneer did not occasionally seem open to criticism to the reader of to-day. But any such criticism would be overwhelmingly outweighed by the simplicity and directness of his methods of attack and his clear grasp of practical issues. His work was intimately related to the historical development of the new statistics.

Americans and others who were not acquainted with Mr. Gosset but who admired "Student" will welcome the Foreword which gives an interesting biographical sketch of this remarkable man and the Frontispiece photograph of him in informal pose.

J. F. KENNEY

An Introduction to Navigation and Nautical Astronomy. By W. G. Shute, W. W. Shirk, G. F. Porter, and Courtenay Hamenway. New York, The Macmillan Company, 1944. 14+457 pages. $4.50.

Certain features of this text are perhaps best described in the words of the authors. The following quotations are taken from the preface:

"This text is written in a language that a beginner in navigation can under-

stand. Technical terms and phrases are not avoided but are carefully defined and explained. Before a new phase of the subject is introduced, preceding material is frequently summarized to show the correlation with what has gone before. Thus the student is constantly reviewing his previous work. Before the subject of celestial navigation is introduced, all necessary basic astronomy is carefully explained, with particular emphasis on its application from the navigator's point of view.

"This text is complete in itself · · · . The H.O. 211 Table is given in full in the Appendix, as well as complete tables of logarithms and trigonometric functions. Excerpts from *The American Nautical Almanac*, *The American Air Almanac*, and the H.O. 214 Tables are sufficient for the solution of all problems given. The student is shown how to work problems of position on ordinary graph paper, thus the difficulty and expense of obtaining special plotting sheets or charts is eliminated. Throughout the text, tables and diagrams from publications of the United States Coast and Geodetic Survey, the Hydrographic Office of the United States Navy, and the United States Coast Guard are reproduced and explained so that the student will become acquainted with these important aids to navigation."

The only solutions of spherical triangles presented in this book are those based on the special tables H.O. 211 and H.O. 214 and the formulas needed in these solutions are not derived, only stated. There is no denying that the exclusive use of these tables permits concentration of effort and consequent mastery of the method given. Many instructors feel, however, that time in the classroom can best be spent mastering underlying principles rather than details of manipulation. Hence, the omission of all derivations in connection with spherical trigonometry is regrettable.

While the tables in this book are adequate to solve all exercises in the text it does materially increase student interest for them to take observations and compute their own positions. This, however, requires current issues of the *Nautical Almanac* which, added to the price of this text, makes it rather expensive for many students.

R. H. BARDELL

NEW BOOKS RECEIVED

Ansayo sobre Algebra de Anillos. By H. E. Calcagno. Buenos Aires, 1942. 28 pages.

Calculus Refresher for Technical Men. By A. A. Klaf. New York, McGraw-Hill Book Company, Inc., 1944. 8+431 pages. $3.00.

The Education of T. C. Mits. By H. G. Lieber and L. R. Lieber. New York, W. W. Norton and Company, Inc., 1944. 230 pages. $2.50.

Graphical Solution. Second Edition. By C. O. Mackey. New York, John Wiley & Sons, Inc.; London, Chapman & Hall, Ltd., 1944. 6+152 pages. $2.50.

Riddles in Mathematics. A Book of Paradoxes. By E. P. Northrop. New York, D. Van Nostrand Company, Inc., 1944. 8+262 pages. $3.00.

DISCUSSIONS AND NOTES

EDITED BY MARIE J. WEISS, Sophie Newcomb College, New Orleans, La.

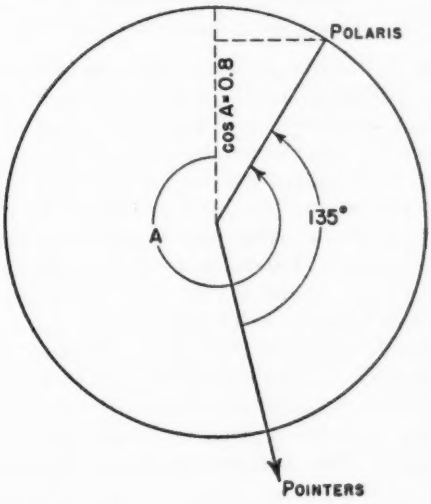
The Department of Discussions and Notes is open to all forms of activity in college mathematics, except for specific problems, especially new problems, which are reserved for the department of Problems and Solutions.

DETERMINATION OF LATITUDE IN AN EMERGENCY

JOEL BRENNER, Lawrence College

This article describes a method for determining latitude quickly and fairly accurately whenever a sextant, Polaris, the "pointers," and the horizon are simultaneously available. No tables or charts are needed.

Theory. The altitude of the elevated north pole is equal to the latitude of the observer. In 1944 Polaris is 1° 00′ from the elevated pole. Hence the latitude of the observer is equal to the altitude of Polaris plus a correction equal to $-60'\cos A$, where A is the hour angle of Polaris (see figure). This angle can be found with sufficient accuracy by adding 138° 30′ to the hour angle of the pointers.



Practice. To find A in the correction formula $60\cos A$, one may estimate the hour angle of the pointers. Woodsmen calculate the time of night from the hour angle of the pointers, and their estimates have an error of less than $30^m = 7°\ 30'$. (This fact suggested the present method to me.) Knowing A, one must find $\cos A$ to one place of decimals. Successful students will find $\cos A$ from its graph. Or, it is easy to find the correction by construction, and without any tables. It is assumed that pencil and paper (or a nail and a piece of wood) are available.

Example. At sea, charts lost, clock stopped, tables illegible, sextant working, navigator takes altitude of Polaris, 44° 30′ 20″, I. C.—0′ 20″, no dip. The refraction correction may be neglected.* After consultation with the crew, it is decided that the figure shows the approximate direction of the pointers. A circle is sketched and drawn free hand (or by the use of two holes in a card, or by the use of a coin); the hour angle of Polaris next estimated (an angle of $135° = 90° + 45°$ is easy to draw); and last the value of $\cos A = 0.8$ can be read to one place of decimals from the figure. The latitude equals $43° 42' \pm 4'$.

If good paper, sharp pencil, and useful work-table are available, and if the sextant circle be used for a protractor, the latitude may be found to 1′ or less. An error in estimating the hour angle of the pointers is least significant when such hour angle is 135° or 315° and most significant when the angle is 225° or 45° ($\cos A$ changes fastest when $A = 90°$ or 270°).

The clock should be started, and a note made of the time and position of the pointers. Difference in longitude made good can be estimated, again by use of the pointers.

A NOTE ON ENVELOPES

HOWARD EVES, Syracuse University

At a certain stage in calculus we teach the well known method of finding the envelope of a one-parameter family of curves given by cartesian equations. Occasionally a student wonders how to find the envelope of a one-parameter family of curves given by polar equations (without transforming to cartesian coördinates and then back again). An easily established but general theorem may be stated to the effect that *the same process used for a family of curves given in the cartesian coördinate system may be used for a family of curves given in any point coördinate system.* Inasmuch as many teachers of college mathematics are unaware of this fact and since it does not seem to be stated in any of the ready literature, it might be well to establish it.

Let $f(u, v, c) = 0$ be the one-parameter family of curves given in some (u, v) coördinate system. Imagine a superimposed (x, y) cartesian coördinate system, and let $u = u(x, y)$, $v = v(x, y)$ be the equations of the transformation of coördinates. Then the family of curves is given in the cartesian system by $F(x, y, c) \equiv f[u(x, y), v(x, y), c] = 0$. Since it is obvious that $\partial F/\partial c \equiv \partial f/\partial c$, the theorem is proved.

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* A good navigator knows the refraction corrections. For the observed altitudes 45°, 30°, 20°, 15°, the corrections are −1′, −2′, −3′, −4′, respectively.

ON THE CONCEPT OF DIVISOR

C. C. MacDuffee, University of Wisconsin

Most books on elementary number theory define divisibility of rational integers as follows:* "An integer a is said to be divisible by another integer b, not 0, if there is a third integer c such that

$$a = b \cdot c.$$"

A perusal of the various books indicates that they fall into two groups: those which specifically put in the condition $b \neq 0$, and those which intend to do so but neglect it. All of them reach the conclusion that two integers not both 0 have a greatest common divisor, and a very few extend the theory by defining the greatest common divisor of 0 and 0 to be 0.

In the theory of ideals it is very clear that the greatest common divisor of 0 and 0 should be 0, and also that the least common multiple of a and 0 should be 0 for every a. It seems as if these conclusions should be obtainable by elementary considerations without the theory of ideals, and this note indicates that only courage to fly in the face of mathematics' most firmly fixed prejudice is required to bring it about.

If three numbers (rational integers) satisfy the relation

$$a = b \cdot c,$$

we shall say that $b \mid a$ (b divides a) and $c \mid a$. There are no reservations. Of course if $b = 0$, c is not uniquely defined so that we do not talk about a quotient in this case, but 0 is divisible by 0.

If $b \mid a$, then a is a *multiple* of b.

THEOREM 1. *Every multiple of* 0 *is* 0.

THEOREM 2. *The number* 0 *is a multiple of every number.*

For $0 = 0 \cdot b$ for every number b.

THEOREM 3. *The only number which is a multiple of every number is* 0.

This is an immediate consequence of Theorems 1 and 2.

A *greatest common divisor* of two numbers a and b is a number d such that

1) $d \mid a$ and $d \mid b$,

2) if $d_1 \mid a$ and $d_1 \mid b$, then $d_1 \mid d$.

A *least common multiple* of two numbers a and b is a number m such that

1) $a \mid m$ and $b \mid m$,

2) if $a \mid m_1$ and $b \mid m_1$, then $m \mid m_1$.

THEOREM 4. *Every two numbers a and b have a greatest common divisor d such that $d = p \cdot a + q \cdot b$, and $d = 0$ if and only if $a = b = 0$.*

Case I, a, b not both 0. That such a d exists and is not 0 is proved in every text-book.

* An Introduction to the Theory of Numbers, G. H. Hardy and E. M. Wright, Oxford University Press, 1938, p. 1.

Case II, $a=b=0$. By Theorem 2, every number is a common divisor of 0 and 0. By Theorem 3, the only multiple of every number is $d=0$. Clearly

$$0 = p\cdot 0 + q\cdot 0$$

for every p and q.

THEOREM 5. *If d and d_1 are two greatest common divisors of a and b, then $d_1=\pm d$.*

The proof of Case I is well known. The proof of Case II is obvious.

THEOREM 6. *Every two numbers a and b have a least common multiple m, and $m=0$ if and only if $ab=0$.*

Case I, $a\neq 0$, $b\neq 0$. It is well known that $m=ab/d$, and that $m\neq 0$.
Case II, $b=0$. The only multiple of 0 is 0, so that $m=0$.

THEOREM 7. $dm=ab$.

If neither a nor b is 0, this is well known. If $ab=0$, then $m=0$ so that it is still satisfied.

THEOREM 8. *The least common multiple of any infinite set of numbers is* 0.

For 0 is a common multiple, and it is the only one, for the set must contain integers of arbitrarily large absolute value.

In line with these considerations is the modern tendency* to define 0 as the order of a group of infinitely many elements. This definition can be justified as follows. Let a be an element of a group G. The period of a may be defined as the non-negative greatest common divisor of all integers n such that $a^n=1$. If there is no such integer n except 0, then clearly this greatest common divisor is 0, and it is natural to define the order of the cyclic subgroup generated by a to be the period of a, namely 0. With the convention that the order of a group of infinitely many elements is 0, it remains true for such groups as well as for finite groups that the order of the group is a multiple of the order of each subgroup, even though there be infinitely many subgroups of different orders, or one of order 0.

When this convention is adopted, the points of view of Steinitz and Albert regarding the characteristic of a field are reconciled. We may modify slightly the Steinitz definition and define the characteristic as the non-negative greatest common divisor d of all rational integers n such that $n\cdot i=0$ where i is the identity element of multiplication in the field. Thus two elements $s\cdot i$ and $t\cdot i$ are equal if and only if s and t are congruent modulo d, congruence modulo 0 being ordinary equality. But the greatest common divisor of all rational integers n such that $n\cdot i=0$ is the order of the (additive) cyclic group generated by the identity element of the field, and this is Albert's definition of characteristic.†

* Zassenhaus, Lehrbuch der Gruppentheorie, Berlin 1937, p. 3.

† A. A. Albert, Modern Higher Algebra, University of Chicago Press, 1937, p. 22.

A VECTORIAL DEVELOPMENT OF TWO DIFFERENTIATION FORMULAS

H. V. CRAIG, University of Texas

Geometrical background. Consider a unit circle with center at the origin. Let s be the arc length measured counterclockwise from the point (1, 0); and let ρ be the radius vector which traces out the circle. Since $\cos s$ and $\sin s$ are the coordinates of the vertex of ρ, we have $\rho = \rho(s) = \cos s\,\mathbf{i} + \sin s\,\mathbf{j}$, $\mathbf{i}$ and $\mathbf{j}$ denoting the unit vectors along the axes. Further, let $\Delta\rho$ be the chord vector $\rho(s+\Delta s) - \rho(s)$. Now since the limit of the ratio chord/arc is unity* and $\rho(s+\Delta s/2)$ is at all times normal to $\Delta\rho$ and approaches $\rho(s)$ as Δs approaches zero; it is tolerably clear that $D\rho$, the limit of $\Delta\rho/\Delta s$, is the unit vector perpendicular to $\rho(s)$ in the sense of increasing s. We are now ready to investigate $D \sin S$ and $D \cos S$.

Proof of the formulas. Having given that $D\rho$ is the unit vector perpendicular to ρ, we see that the slope of $D\rho$ is $-\cos s/\sin s$ and finally that $D\rho = -\sin s\,\mathbf{i} + \cos s\,\mathbf{j}$. But by the rules of elementary calculus, and they obviously apply in the present case, $D\rho = D\cos s\,\mathbf{i} + D\sin s\,\mathbf{j}$ and by equating corresponding components $D\cos s = -\sin s$, $D\sin s = \cos s$.

PROBLEMS AND SOLUTIONS

ELEMENTARY PROBLEMS

EDITED BY OTTO DUNKEL, ORRIN FRINK, JR., AND H. S. M. COXETER

Send all communications concerning Elementary Problems and Solutions to H. S. M. Coxeter, 24 Southearn Boulevard, Toronto 10, Canada.

The department of Elementary Problems welcomes problems believed to be new, and demanding no tools beyond those ordinarily furnished in the first two years of college mathematics. To facilitate their consideration, solutions should be submitted on separate, signed sheets, within three months after publication of problems.

PROBLEMS FOR SOLUTION

E 626. *Proposed by W. E. Buker, Perry High School, Pittsburgh*

A ship at sea attempts to determine its position by plotting lines of position through observation of three stars. Due to inaccuracy of observation, these lines are not concurrent but form a triangle. What is the most probable position of the ship, assuming (a) that the errors are due to defective instruments and are of the same kind, (b) that the errors are due to inaccurate observation and may not all be of the same kind?

E 627. *Proposed by V. Thébault, San Sebastián, Spain*

Construct a convex enneagon, given the centers of exterior squares described on the sides.

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* For an elementary argument, see M. S. Knebelman, An elementary limit, this MONTHLY, vol. 50, p. 507.

E 628. *Proposed by W. C. Rufus, Observatory of the University of Michigan*

Find the smallest positive integer, one-half of which is a square, one-third of which is a cube, and one-fifth of which is a fifth power.

E 629. *Proposed by H. S. M. Coxeter, University of Toronto*

Sum the series

$$\sum_{r=0}^{[n/2]} \left(-\frac{1}{4}\right)^r \binom{n-r}{r}.$$

E 630. *Proposed by R. A. Rosenbaum, U.S.N.R.*

For any point P of a given closed convex curve C, let P' be that point on the exterior normal to C at P for which $PP'=d$, a constant. The locus of P' is a curve C', parallel to C. Let s, s' be the respective lengths of C, C', and A, A' the areas within these curves. Show that

$$s' = s + \pi d,$$
$$A' = A + sd + \pi d^2.$$

(Cf. 4036 [1943, 397].)

SOLUTIONS

Intersections of Orthogonal Circles

E 591 [1943, 560]. *Proposed by C. J. Coe, University of Michigan*

Two given coplanar circles, (A_1) and (A_2), are cut orthogonally by a third circle, (B). Prove that a line joining either intersection point on (A_1) to either intersection point on (A_2) will pass through one of two points on the line of centers A_1A_2, these two points being the same for all choices of the orthogonal circle (B).

I. *Solution by J. S. Guérin, Catholic University of America.* The following equivalent theorem will be proved. If tangents be drawn to two circles (A_1) and (A_2) from any point on their radical axis, the line joining either point of contact on (A_1) to either point of contact on (A_2) passes through one of the centers of similitude of (A_1) and (A_2).

Let C_1 on (A_1) and C_2 on (A_2) be two such points of contact. If the line C_1C_2 intersects (A_2) in a second point D_2, the triangle $C_2A_2D_2$ is isosceles, and

$$\angle A_2D_2C_2 = \angle A_2C_2D_2.$$

But, since B lies on the radical axis of the two circles, the triangle BC_1C_2 is isosceles; thus

$$\angle BC_1C_1 = \angle BC_2C_1 = \tfrac{1}{2}\pi - \angle A_2C_2D_2 = \tfrac{1}{2}\pi - \angle A_2D_2C_2,$$

and

$$\angle BC_1C_2 + \angle A_2D_2C_2 = \tfrac{1}{2}\pi.$$

Hence A_2D_2 is perpendicular to BC_1, and the radii A_1C_1, A_2D_2 are parallel. In other words, the line $C_1C_2D_2$ passes through a center of similitude of the circles.

II. *Solution by Augustus Sisk, Maryville, Tenn.* (Numbers in parentheses refer to articles in N. Altshiller Court's *College Geometry*.) By (325), B lies on the radical axis of (A_1) and (A_2). By (331), the points of contact of two tangents drawn from B to the respective circles are antihomologous points. But lines joining antihomologous points on two circles pass through one of the centers of similitude of the circles.

Also solved by Howard Eves and L. M. Kelly.

Simultaneous Equations

E 594 [1943, 560]. *Proposed by F. J. Duarte, Caracas, Venezuela*

Solve the equations

$$x^2 = y^3, \qquad (x - 3)y + (1 - a - b)(3y - 2x) = 0,$$

where

$$ab = -1, \qquad a^3 + b^3 = 1.$$

Solution by E. P. Starke, Rutgers University. For simplicity put $s=a+b$. Then

$$s(s^2 + 3) = s(s^2 - 3ab) = (a + b)(a^2 - ab + b^2) = a^3 + b^3 = 1.$$

That is, s is a root of $s^3+3s-1=0$, or $(s-1)^3+3s^2=0$. From the second given equation we get $y=2x(s-1)/(3s-x)$, which gives, upon substitution in the first equation,

$$8x^3(s - 1)^3/(3s - x)^3 = x^2.$$

Hence $x=0$ (with $y=0$) is a double solution. If $x\neq 0$, we have

$$(3s - x)^3 = 8x(s - 1)^3 = -24xs^2,$$

whence

$$(x - 9s)(x^2 + 3s^2) = 0.$$

We thus obtain the solutions

$$\begin{aligned} x &= 9s, & y &= 3(1 - s); \\ x &= is\sqrt{3}, & y &= \omega(s - 1); \\ x &= -is\sqrt{3}, & y &= \omega^2(s - 1); \end{aligned}$$

where $\omega=e^{2\pi i/3}$, $s=a-a^{-1}$, and $a^3=(\sqrt{5}+1)/2$.

Also solved by Hazel Schoonmaker Wilson and the proposer.

An Almost Equiangular Integral Triangle

E 595 [1943, 560]. *Proposed by H. T. R. Aude, Colgate University*

Find the smallest set of three different integers to represent the sides of a

triangle in which one angle is 60° and each of the other angles differs therefrom by not more than one minute.

Solution by C. L. Weaver, Boston, Mass. Let the sides of the triangle be $a<b<c$. Put $a=b-x$, $c=b+y$. By the law of cosines,

$$b^2 = a^2 + c^2 - ac,$$

or

$$b = \frac{x^2 + y^2 + xy}{x - y} = \frac{3y^2 + 3yz + z^2}{z}, \qquad z = x - y > 0.$$

By the law of sines,

$$\frac{b}{y} > \frac{\sin 60°}{\sin 60° 1' - \sin 60°} = 5955.85.$$

Substituting the above expression for b, this gives

$$\left(\frac{y}{z}\right)^2 - 1984.28\frac{y}{z} + \frac{1}{3} > 0.$$

Therefore either $y/z>1984$ or $y/z<1/5952$. The latter possibility is ruled out since $3y^2$ must be a multiple of the integer z. Hence the least solution in positive integers is

$$z = 1, \qquad y = 1985, \qquad x = 1986,$$

whence

$$a = 11,824,645, \qquad b = 11,826,631, \qquad c = 11,828,616.$$

Also solved by R. A. Johnson, Joseph Rosenbaum, and Howard Eves.

Concurrent Circles

E 597 [1943, 633]. *Proposed by V. Thébault, San Sebastián, Spain*

Let P be any point in the plane of a triangle ABC. Let (U), (V), (W) denote the circles BCP, CAP, ABP, and (U'), (V'), (W') their images by reflection in the respective sides BC, CA, AB. Also let u, u' be the powers of A with respect to (U), (U'), and let v, v', w, w' be defined analogously. Show that the circles (U'), (V'), (W') are concurrent, and that

$$u + u' + v + v' + w + w' = a^2 + b^2 + c^2.$$

Solution by J. S. Guérin, Catholic University of America. Let P' denote the second point of intersection of (V') and (W'). Since

$$\sphericalangle APB = \sphericalangle AP'B \quad \text{and} \quad \sphericalangle APC = \sphericalangle AP'C,$$

it may readily be shown that $\sphericalangle BPC = \sphericalangle BP'C$. It follows that P' is on (U'). Thus (U'), (V'), (W') are concurrent in P'.

Let M denote the midpoint of BC, m_a the median AM, and R_a the radius of (U). We have

$$u + u' = AU^2 + AU'^2 - 2R_u^2 = 2AM^2 + 2UM^2 - 2R_u^2$$
$$= 2m_a^2 + 2(R_u^2 - BC^2/4) - 2R_u^2 = 2m_a^2 - a^2/2.$$

Similarly, $v+v' = 2m_b^2 - b^2/2$ and $w+w' = 2m_c^2 - c^2/2$. Hence

$$u + u' + v + v' + w + w' = 2(m_a^2 + m_b^2 + m_c^2) - (a^2 + b^2 + c^2)/2$$
$$= a^2 + b^2 + c^2.$$

Also solved by the proposer.

A Class of Convergent Sequences

E 598 [1943, 633]. *Proposed by H. S. Wall, Northwestern University*

Let $g_1, g_2, g_3, \cdots$ be any numbers such that

$$0 < g_p < 1, \qquad (1 - g_p)g_{p+1} > \tfrac{1}{4} \qquad (p = 1, 2, 3, \cdots).$$

Prove that

$$\lim_{p\to\infty} g_p = \tfrac{1}{2}.$$

Solution by Irving Kaplansky, Harvard University. For any real number x we have $x(1-x) \leqq \frac{1}{4}$. Hence

$$(1 - g_p)g_{p+1} > (1 - g_p)g_p,$$

i.e., the g's are monotone increasing. Since they are bounded they approach a limit z which must satisfy $(1-z)z \geqq \frac{1}{4}$. Hence $z = \frac{1}{2}$.

More generally, suppose $0 < g_p < 1$ and $(1-g_p)g_{p+1} > a > 0$. Then $a \leqq \frac{1}{4}$ and

$$\underline{\lim}\, g_p \leqq (1 + \sqrt{1 - 4a})/2, \qquad \overline{\lim}\, g_p \geqq (1 - \sqrt{1 - 4a}\,)/2.$$

For, defining the sequence $\{x_p\}$ by

$$x_1 = 0, \qquad x_{p+1} = a/(1 - x_p) \qquad (p = 1, 2, \cdots),$$

so that $\lim x_p = (1 - \sqrt{1-4a})/2$, we easily prove by induction that

$$x_p < g_p < (1 + \sqrt{1 - 4a})/2.$$

In the extreme case when $a = \frac{1}{4}$, we find $x_p = (p-1)/2p$, so

$$(p - 1)/2p < g_p < \tfrac{1}{2}.$$

Also solved by J. M. Danskin, Jr., F. J. Duarte, N. J. Fine, J. B. Kelly, W. H. Myers, E. A. Nordhaus, William Scott, M. F. Smiley, E. P. Starke, J. E. Wilkins, Jr., and the proposer. One would-be solver believed he had disproved the existence of the proposer's sequence. But all the conditions are satisfied by the instance

$$g_p = (kp - 1)/2kp,$$

for any $k > 1$.

ADVANCED PROBLEMS

Send all communications about Advanced Problems and Solutions to Otto Dunkel, Washington University, St. Louis, Mo. All manuscripts should be typewritten, with double spacing and with margins at least one inch wide.

Problems containing results believed to be new or extensions of old results are especially sought. The editorial work would be greatly facilitated, if, on sending in problems, proposers would also enclose any solutions or information that will assist the editors in checking the statements. In general, problems in well known text-books or results found in readily accessible sources will not be proposed as problems for solution in this department. In so far as possible, however, the editors will be glad to assist members of the Association in the solution of such problems.

PROBLEMS FOR SOLUTION

4124. *Proposed by T. W. Anderson, Jr., Princeton University*

Consider the set of n by n matrices whose entries are positive integers or zero. Let the sum of the entries of the ith row be r_i, $i=1, 2, \cdots, n$, and the sum of the entries of the jth column be c_j, $j=1, 2, \cdots, n$. For specified r_i and c_j, positive or zero integers, with

$$\sum_{i=1}^{n} r_i = \sum_{j=1}^{n} c_j$$

what are the minimum and maximum sums of entries in the main diagonal, *i.e.*, the minimum and maximum traces?

4125. *Proposed by Hüseyin Demir, Columbia University*

Prove that

$$\begin{vmatrix} \sin\theta_1 & -e^{-i\theta_1} & 0 & 0 & \cdots & 0 & 0 \\ \sin\theta_2 & e^{i\theta_2} & -e^{-i\theta_2} & 0 & \cdots & 0 & 0 \\ \sin\theta_3 & 0 & e^{i\theta_3} & -e^{-i\theta_3} & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & & \vdots & \vdots \\ \sin\theta_n & 0 & 0 & 0 & \cdots & 0 & e^{i\theta_n} \end{vmatrix} = \sin(\theta_1+\theta_2+\cdots+\theta_n).$$

4126. *Proposed by A. D. Wallace, University of Pennsylvania*

Let x, A, b denote respectively $(1, m)$, (m, n), $(1, n)$ matrices, an (i, j) matrix being one with i rows and j columns. If the matrix AA' is non-singular, show that the least square solution of $xA=b$ is the solution of an equation of the form $xA=b_0$. Determine b_0 and its geometric meaning.

4127. *Proposed by V. Thébault, San Sebastián, Spain*

The straight lines AG, BG, CG, DG drawn through the vertices and centroid G of the tetrahedron $ABCD$ meet its circumsphere again in A_1, B_1, C_1, D_1, and the planes perpendicular to the respective lines at these latter points determine the tetrahedron $A_2B_2C_2D_2$.

Show that (1). The two tetrahedrons have the same centroid and are hyperbolic. (2). The non-focal axis of the quadric surface with G as a focus inscribed in $A_2B_2C_2D_2$ is equal to the diameter of the Monge sphere of the Steiner ellipsoid inscribed in $ABCD$.

Note. The first part extends to the tetrahedron the properties of the triangle indicated by W. F. Beard, *Educational Times, Reprints*, 1917, p. 69.

SOLUTIONS

An Algebraic Identity and Partitions

4067 [1943, 65]. *Proposed by H. S. Wall, Northwestern University*

Prove that, if $0<x<1$,

$$\prod_{n=1}^{\infty}(1-x^{2n-1}) = 1\Big/\left[1+\sum_{k=1}^{\infty}x^{k(k+1)/2}/(1-x)(1-x^2)(1-x^3)\cdots(1-x^k)\right].$$

I. *Solution by R. C. Buck, Cambridge, Mass.* Consider the Euler identity

$$(1+x^2)(1+x^4)(1+x^6)\cdots = 1+\frac{x^2}{1-x^2}+\frac{x^2}{1-x^2}\frac{x^4}{1-x^4}+\frac{x^2}{1-x^2}\frac{x^4}{1-x^4}\frac{x^6}{1-x^6}+\cdots. \tag{1}$$

(Hardy and Wright, *Theory of Numbers*, p. 275, Theorem 346, or p. 278, Theorem 348.) This holds for $-1<x<1$, hence for $0<x^2<1$. Replace x^2 by x, and we have

$$(1+x)(1+x^2)(1+x^3)(1+x^4)\cdots = 1+\frac{x}{1-x}+\frac{x}{1-x}\frac{x^2}{1-x^2}+\frac{x}{1-x}\frac{x^2}{1-x^2}\frac{x^3}{1-x^3}+\cdots. \tag{2}$$

Now,

$$(1+x)(1+x^2)\cdots = \frac{1-x^2}{1-x}\frac{1-x^4}{1-x^2}\cdots = \frac{1}{1-x}\frac{1}{1-x^3}\frac{1}{1-x^5}\cdots$$

and combining this with (2), we have

$$(1-x)(1-x^3)(1-x^5)\cdots = \frac{1}{1+\dfrac{x}{1-x}+\dfrac{x}{1-x}\dfrac{x^2}{1-x^2}+\cdots}$$

which is the desired result.

II. *Solution by D. H. Lehmer, University of California.* This identity was known to Euler who gave an algebraic proof of it. The following "graphical" proof may be of interest. The proposed identity may be written

$$\frac{1}{(1-x)(1-x^3)(1-x^5)\cdots} = \sum_{k=0}^{\infty} \frac{x^{k(k+1)/2}}{(1-x)(1-x^2)\cdots(1-x^k)}.$$

Comparing the coefficient of x^n $(n>0)$ in the power series expansion of each side we see that the identity is equivalent to the following:

THEOREM A. *The number of partitions of n into odd parts is the sum, for all possible values of k, of the numbers of partitions of $n-(k+1)k/2$ into parts not exceeding k.*

This theorem follows at once from the following two theorems:

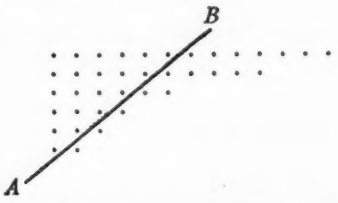
THEOREM B. *The number of partitions of n into odd parts is equal to the number of partitions of n into distinct parts.*

THEOREM C. *The number of partitions of $n-(k+1)k/2$ into parts $\leqq k$ is equal to the number of partitions of n into precisely k distinct parts.*

Theorem C may be proved graphically as follows. A partition, such as

$$13+10+6+4+3+2=38$$

may be represented graphically by the diagram



when it is read by rows. It is clear that partitions of n into precisely k distinct parts correspond to graphs of k rows, each of lesser length than row above it. Given such a graph we construct the line AB running in a northeasterly direction and passing just south of the first point of the last row. This line divides the graph into a "head" containing $(k+1)k/2$ points and a "tail." This tail, if read by diagonals parallel to AB, clearly gives a partition of $n-(k+1)k/2$ into parts $\leqq k$. In the example $k=6$ and the tail gives the partition

$$6+3+2+2+2+1+1=17$$

of $17=38-(6+1)6/2$ into parts $\leqq 6$. Hence to each partition of n into k distinct parts there corresponds a partition of $n-(k+1)k/2$ into parts $\leqq k$. To show that this correspondence is one to one, let any partition of the latter type be given. We construct a graph which, when read by columns, gives this partition. Then no row will contain more points than the row above it. We now shift the rth row bodily r units to the left $(r=1, 2, \cdots, k)$. The graph is now a tail to which we fix a head of $(k+1)k/2$ points to obtain a regular graph which,

when read by rows, gives a partition of n into k necessarily distinct parts. Hence the correspondence is one to one and Theorem C follows.

Theorem B is of course very well known. A graphical proof of it, not so well known, is due to Sylvester. It consists in reading in two ways a graph which is symmetrical with respect to its main diagonal. The two ways are indicated by the following figures.

$9+9+5+5+3+1+1=33$

$11+8+7+5+2=33$

The correspondence thus established is easily seen to be one-to-one.

A Minimum Given by the First Brocard Triangle

4074 [1943, 125]. *Proposed by V. Thébault, San Sebastián, Spain*

On the sides of the given triangle ABC directly similar triangles ABC_1, etc., are constructed interiorly. Determine these latter triangles so that $(GA_1)^2+(GB_1)^2+(GC_1)^2$ is a minimum, where G is the centroid of ABC.

Editorial Note. The proposer gave the following remarks. The sum of the squares of the sides of triangle $A_1B_1C_1$ is equal to three times the sum of the squares of the distances of G to its vertices, so that it suffices to make the first sum a minimum. The vertices A_1, B_1, C_1 which give the desired minimum are the vertices of the first Brocard triangle for ABC, for which

$$\frac{\overline{B_1C_1}^2+\overline{C_1A_1}^2+\overline{A_1B_1}^2}{\overline{BC}^2+\overline{CA}^2+\overline{AB}^2}=\frac{1}{4}(1-3\tan^2 V),$$

where V is the Brocard angle for ABC.

A proof may be obtained by taking the centroid G as the origin for the complex coordinates t_1, t_2, t_3; u_1, u_2, u_3 of A, B, C; A_1, B_1, C_1 and setting $\lambda=e^{i\theta}$, $\theta=\angle BAC_1$, $0\leqq\theta<\pi$, and $r=AC_1/AB$. Then $u_3=(1-r\lambda)t_1+r\lambda t_2$, *etc.*, and we see at once that G is also the centroid of $A_1B_1C_1$. Denote by l_1 and m_1 the lengths of GA and GA_1. It then follows that

$$\sum m_j^2=(1-2r\cos\theta+2r^2)\sum l_j^2+r\sum(\lambda t_2\bar{t}_1+\bar{\lambda}t_1\bar{t}_2)-r^2\sum(t_2\bar{t}_1+t_1\bar{t}_2).$$

This may be transformed by using

$$\sum(t_2\bar{t}_1+t_1\bar{t}_2)=-\sum l_j^2=2\sum l_1l_2\cos(\theta_2-\theta_1),\qquad t_j=l_je^{i\theta_j},$$

$$\sum(\lambda t_2\bar{t}_1+\bar{\lambda}t_1\bar{t}_2)=-\cos\theta\sum l_j^2-4\sin\theta S,\quad S=\text{area } ABC,\quad \tan V=4S/3\sum l_j^2.$$

We then have

$$\frac{\sum m_i^2}{\sum l_i^2} = 3r^2 - \frac{3r\cos(\theta - V)}{\cos V} + 1$$

and from this we find that the minimum is given by $\theta = V$, $2r\cos V = 1$. Thus the triangles ABC_1, *etc.*, which give the minimum are isosceles with base angle V and the triangle $A_1B_1C_1$ for the minimum is therefore the first Brocard triangle for ABC.

A Transcendental Number

4078 [1943, 263]. *Proposed by Raphael Robinson, Univ. of California at Berkeley*

Show that the continued fraction

$$\frac{1}{2^{1!}} + \frac{1}{2^{2!}} + \frac{1}{2^{3!}} + \frac{1}{2^{4!}} + \cdots,$$

represents a transcendental number by using the following theorem of Liouville, *Journal de Mathématiques pures et appliquées*, 1851.

The partial quotients of a continued fraction representing the root x of an nth degree algebraic equation with rational coefficients never exceed the product of a certain constant by the $(n-2)$nd power of the denominator of the preceding convergent.

Solution by the Proposer. Let the kth convergent of the continued fraction

$$b_0 + \frac{1}{b_1} + \frac{1}{b_2} + \frac{1}{b_3} + \cdots$$

be A_k/B_k, where $b_0/1$ is considered as the zeroth convergent. Then $B_k = b_kB_{k-1} + B_{k-2} < (b_k+1)B_{k-1}$. By repeated application of this formula one obtains, since $B_0 = 1$,

$$B_k < (b_1 + 1)(b_2 + 1)\cdots(b_k + 1).$$

For the fraction in this problem

$$B_k < (2^{1!} + 1)(2^{2!} + 1)\cdots(2^{k!} + 1) < 2^{1!+1}\,2^{2!+1}\cdots 2^{k!+1}$$
$$= 2^{k+1!+2!+\cdots+k!} < 2^{2k!}$$

if $k > 3$, since $k+1!+2!+\cdots+(k-1)!$ is the sum of k terms, none exceeding $(k-1)!$, and not all this great, and hence less than $k!$. Therefore $B_{2\lambda-1} < 2^{2(2\lambda-1)!}$, so that

$$b_{2\lambda} = 2^{(2\lambda)!} = 2^{2\lambda(2\lambda-1)!} > B_{2\lambda-1}^{\lambda} = B_{\lambda-1}^{\lambda-n+2} B_{2\lambda-1}^{n-2}$$

and $B_{2\lambda-1}^{\lambda-n+2}$ increases with λ beyond all limits, no matter what the value of n, so that the continued fraction does not satisfy an equation of the nth degree for any value of n, *i.e.*, is transcendental.

Editorial Note. The Liouville theorem stated in the problem results easily from the final inequality in the proof of Theorem 191, p. 160, in Hardy and Wright's *An Introduction to the Theory of Numbers.* Following this proof are two examples of transcendental numbers

$$(a) \qquad \sum_{i=1}^{\infty} 10^{-i!}, \qquad \frac{1}{10} + \frac{1}{10^{2!}} + \frac{1}{10^{3!}} + \cdots;$$

and it is stated that it is obvious that the base 10 may be replaced by other integers. The above problem was received before the appearance of this text in print.

NEWS AND NOTICES

Readers are invited to contribute to the general interest of this department by sending news items to B. W. Jones, White Hall, Cornell University, Ithaca, New York.

Rutgers University has announced the creation of a Research Council with funds to strengthen and promote research throughout the university. Emphasis will be placed upon cooperative research between related departments and between outside organizations and university departments.

Professors G. D. Birkhoff and Harlow Shapley of Harvard University have been elected honorary members of the Mexican Mathematical Society. The former has also been elected corresponding member of the Mexican National Academy of Sciences.

Associate Professor Saunders Mac Lane of Harvard University has been elected a member of the American Academy of Arts and Sciences.

Assistant Professor A. E. Taylor and Professor W. M. Whyburn of the University of California at Los Angeles have been elected corresponding members of La Academia Nacional de Ciencias Exactas, Físicas y Naturales de Lima.

Associate Professor A. L. Underhill of the University of Minnesota represented the Mathematical Association at the inauguration of V. J. Flynn as president of the College of Saint Thomas on April 27, 1944.

The following promotions at Alfred University are announced: W. V. Nevins, III, to an assistant professorship, Assistant Professor L. R. Polan to an associate professorship, and Associate Professor L. L. Lowenstein to a professorship.

Dr. Leon Alaoglu, Professor W. L. Ayres and Assistant Professor J. W. T. Youngs of Purdue University are serving with the Army Air Forces. Professor Laurence Hadley is acting head of the department of mathematics.

Assistant Professor K. J. Arnold of the University of New Hampshire is now serving with the Statistical Research Group at Columbia University.

F. L. Celauro of Syracuse University has been appointed mathematician in the National Bureau of Standards in Washington.

Assistant Professor G. M. Ewing of the University of Missouri has been granted a leave of absence to serve in the Naval Ordnance Laboratory in Washington.

Professor F. A. Ficken of the University of Tennessee has been granted a leave of absence to serve in the Office of Scientific Research and Development in Washington.

N. J. Fine of Purdue University is now in the Research laboratory of the Lukas-Harold Corporation in Indianapolis.

Dr. R. H. Fox of the University of Illinois has been appointed to an assistant professorship at Syracuse University.

Dr. Michael Golomb of Purdue University has been promoted to an assistant professorship and is on leave for war research at Franklin Institute.

A. W. Goodman of Syracuse University is now serving with the Republic Aviation Corporation in Farmingdale, New York.

Dr. W. H. Gottschalk of the University of Virginia has been appointed to an assistant professorship at the University of Richmond.

Frank Hawthorne of Alliance College has been granted a leave of absence to serve as an associate physicist in the University of California Division of War Research in New York City.

Professor L. A. Hazeltine of Stevens Institute of Technology has resigned.

Associate Professor B. W. Jones of Cornell University has been promoted to a professorship.

Margaret E. Jones of Ohio State University has been promoted to an assistant professorship.

Dr. G. K. Kalisch of the University of Kansas has been appointed to an instructorship at Cornell University.

Associate Professor P. W. Ketchum of the University of Illinois has been given a leave of absence to engage in war research at Columbia University.

Professor J. F. Locke of Memphis State College has been granted a leave of absence to serve at the United States Naval Academy.

H. J. Miser of Illinois Institute of Technology has been appointed lecturer at Lawrence College, Appleton, Wisconsin.

Associate Professor L. F. Ollman of Hofstra College has been promoted to a professorship.

M. M. Resnikoff has been appointed to a professorship at State Teachers College, Minot, North Dakota.

Assistant Professor H. M. Schwartz of the University of Idaho has been appointed research fellow at the Bartol Research Foundation in Swarthmore, Pennsylvania.

Dr. M. E. Shanks of the University of Missouri has been granted a leave of absence to serve as a mathematician in the Aircraft Engine Research Laboratory at Cleveland.

Professor I. S. Sokolnikoff of the University of Wisconsin has been granted leave of absence to engage in war research.

Assistant Professor C. C. Torrance of Case School of Applied Science is on leave of absence serving with the Bureau of Ordnance of the Navy Department at Washington, D. C.

Professor H. S. Wall of Northwestern University has been appointed Visiting Lecturer at Illinois Institute of Technology.

A. H. Wheeler has been appointed to an affiliate professorship in the engineering staff for the A.S.T.P. program at Clark University.

Associate Professor A. S. Winsor of the University of North Carolina has been promoted to a professorship.

Dr. Clyde Wolfe of the Radiation Laboratory of the University of California died March 25, 1944.

WAR INFORMATION

EDITED BY C. V. NEWSOM

Send news reports upon the utilization of mathematicians or mathematics in war activities to C. V. Newsom, Oberlin College, Oberlin, Ohio.

THE TEACHER PROBLEM IN SECONDARY MATHEMATICS

Probably one of the most serious effects of the war on American education has been the withdrawal of approximately one-third of the trained teachers in the public schools to go into other fields of work. As a result, students below the

college level are being taught in many instances by persons who have the minimum qualifications to obtain emergency certificates. This means that a very large number of children are now receiving very poor instruction.

In many of the subject-matter courses of high school, the shortage of trained teachers has been partially offset by the tremendous decrease in the number of high school students. According to Circular No. 227 of the U. S. Office of Education, there were approximately 1,700,000 fewer students in high school in 1943 who were fifteen years of age and over than there were in 1940. In spite of this decrease, however, the actual number of students studying mathematics and physics is greater now than in previous years. In fact, the number of students registered in physics increased 11.6% from 1942 to 1943. In mathematics, the only other subject to show a positive increase, there were 3.5% more students in 1943 than in 1942. Such special courses as pre-induction mathematics and pre-flight mathematics have augmented the registration in mathematics. Of the special war-time courses upon a secondary level suggested by the Army and the U. S. Office of Education, it is interesting to observe that the largest enrollment of girls is to be found in pre-induction mathematics.

So the number of teachers in mathematics and physics now needed in high school is generally greater at the present time than in pre-war years. But, such teachers are continually receiving offers from governmental agencies and war industries. Teacher-training institutions have found it impossible to fill the rapidly increasing gap between the supply and the demand in view of the fact that they have suffered such a great decrease in enrollment. This latter situation in itself is critical since the supply of new well-trained teachers will be very small for a number of years. The U. S. Office of Education has made the comment that "the period of poor teaching will be prolonged after the war, until the returning teachers can bring their training up to date and normal size classes are graduated from teacher-training institutions."

The Engineering, Science, and Management War Training Program of the U. S. Office of Education has attempted to assist in the solution of the problem by sponsoring free, non-credit correspondence courses in mathematics and physics for those who might become high school teachers. These courses are being offered co-operatively by 19 colleges and universities. The response to this program up to the present time has been small, perhaps due to limited publicity.

A prospectus distributed by representatives of the ESMWT Program explaining the correspondence courses in physics and mathematics emphasizes that "the shortage of knowledge in mathematics and physics is a war emergency." The seriousness of the lack of competent instructors is portrayed in the following paragraph.

"The demand for persons capable of teaching high-school mathematics and physics has increased faster than the supply has been replenished. Both the armed forces and war industry require some competence in these fields, in consequence more young men and women than ever before must be taught mathematics and physics, especially in the high schools. The supply of persons capable of

teaching high-school mathematics and physics has been dwindling since before America's entry into the war. They have been drawn off into the services, industry, and war research. These are not tenuous estimates, but facts. They are of critical importance. Between the multiplying demand for instruction and the shrinking supply of instructors, particularly in smaller communities, there is a gap—a shortage potentially more serious than those in rubber, steel, and other strategic materials."

To register for an ESMWT correspondence course in mathematics, the registrant must indicate that he has completed high school or college courses in algebra and geometry. Moreover, he must obtain a statement from a responsible school official that he is now employed as a teacher of mathematics or physics, or that he will have reasonable opportunity for such employment upon the completion of the course. There are no fees of any kind, except that every student is expected to purchase his own textbooks. No academic credit is given for the work.

The course in mathematics, as outlined, includes topics in arithmetic and percentage, experimental geometry, intermediate algebra, logarithms, plane trigonometry, and spherical trigonometry. There are 48 lessons or assignments, the equivalent of six semester hours, or approximately nine quarter hours. It is expected that the material in the course can be covered in twenty-six weeks or less.

Persons desiring more information in regard to the ESMWT courses should apply to the ESMWT institutional representative in one of the following institutions: The Universities of Alabama, Arkansas, California, Chicago, Florida, Indiana, Iowa, Kansas, Louisiana, Minnesota, Nebraska, North Carolina, Tennessee, Texas, Utah, Washington, Wisconsin, or the State College of Oregon or Pennsylvania.

EASTERN CONFERENCE ON NAVY V-12 PROGRAMS

A conference attended by 32 representatives of 28 Eastern colleges and universities which have Navy V-12 Programs was held January 22, 1944. The meeting was sponsored by the American Council on Education. Certain items from the report of the conference appear below.

1. Commander Alvin C. Eurich, Officer in Charge of the Standards and Curriculum Section, Bureau of Naval Personnel, indicated that the Navy is not interested in the development of programs leading to degrees, but that the matter of degrees is the responsibility of the institutions. He referred to the statement on page 6 of the *Navy V-12 Bulletin No. 101*, concerning this matter. He indicated his hope, however, that the institutions with V-12 units might develop a fairly uniform policy with respect to the award of degrees for the completion of the Navy programs.

2. The conference unanimously adopted the following resolution:

"Resolved that it is the opinion of this group that a student who has satisfactorily completed the Navy V-12 curricula prescribed for deck candidates

should be regarded as having completed the equivalent of the four terms of the normal freshman and sophomore studies leading to the Bachelor's Degree. The college in which he may then seek enrollment should so adjust the curriculum of the junior and senior years that the student can meet the degree-requirements of the institution, with such modifications as may be necessary, within the limit of four additional terms.

"None of the prescribed courses of the Navy curriculum should be regarded as unacceptable for credit merely on the ground that the subject is not normally included within the institution's regular offerings."

3. After some discussion of required calculus in the third and fourth terms of the curriculum for deck candidates, the conference unanimously adopted the following resolution:

"That we request the Training Division of the Bureau of Naval Personnel to re-examine the curricula for deck candidates in the third and fourth terms, looking toward the elimination of required calculus, and to provide more effectively for differences in aptitude."

SOME DATA ON MATHEMATICS IN COLLEGES

Circular No. 228 of the U. S. Office of Education, shortly to be published, contains a statistical study of the effects of the war upon the colleges for the period, October 15, 1942–October 15, 1943. Circular No. 217 considered the period, October 15, 1941–October 15, 1942. The following material is from Circular No. 228. Figures on student registrations refer to nonmilitary enrollments estimated on the basis of returns from 561 institutions. The student tabulation under "all fields" covers engineering, biology, chemistry, economics, home economics, mathematics, physics and sociology.

Item	*All fields*		*Mathematics*	
	Men	*Women*	*Men*	*Women*
Teachers leaving their positions during 1943	8,239	4,293	442	145
Positions still unfilled on Oct. 15, 1943	1,846	785	110	120
Students on Oct. 15, 1943:				
Juniors	12,491	25,323	345	1,896
Seniors	13,119	22,982	363	1,538
Graduates	6,233	2,735	314	205
Students on Oct. 15, 1942:				
Juniors	40,310	21,190	2,065	1,412
Seniors	33,372	19,523	1,681	1,242
Graduates	6,313	3,976	465	255
Degrees conferred, 1942–'43:				
Bachelor's	92,273	93,467	1,071	1,277
Master's	9,987	7,840	68	74
Doctor's	3,025	429	42	9

MATHEMATICAL PUBLICATION DURING WARTIME

Available information indicates that the publication of mathematical research is generally being maintained at a high level throughout the world in spite of war-time handicaps. In this country, the quantity and the quality of the work being published is definitely comparable with that of the pre-war period. Editors of some of the journals have indicated, however, that a noticeable decrease in quantity may be expected in the near future. Most of the journals in this country can now publish acceptable papers with a minimum of delay.

As for foreign countries, the decrease in the amount of publication has, of course, been more radical. There are some interesting developments, however, which are not generally known in this country. For instance, the Russian mathematicians are very active, and are doing most outstanding work. It should be understood that this work is in pure mathematics to a very large extent, and even in its most abstract branches like number theory. The Stalin prize has repeatedly been given for such work. The Russians are also doing much in applied mathematics, but Russian applied mathematics has a character of its own. The Russians seem able to apply the most abstruse mathematical methods with great success, whereas in other countries, the so-called applied mathematics is often more in the field of engineering than in mathematics. Papers of an exceedingly mathematical character appear in Russian journals devoted to geophysics, engineering, and so on.

The mathematical work being done by some excellent Chinese mathematicians is outstanding. The new Chinese mathematics is exceedingly abstract, and is done under the most unfavorable circumstances. One of the younger Chinese mathematicians (S. S. Chern) has recently arrived in Princeton, and there is hope that some more will arrive.

In France, the mathematical production appears to have been reduced about 50%. The *Comptes Rendus* of Paris appear quite regularly; the same is true of some other French journals like the *Journal de Mathematiques*.

Mathematical production in Germany seems to be quite normal, and not very much reduced. In the case of Italy, very little information is available in regard to research activities.

The British mathematical output has been quite steady throughout the years of the war. Moreover, there appears to have been no essential change in its character.

NOTES ON POST-WAR EDUCATION

At present writing, Congress is still debating measures which provide for educational benefits for veterans. The Thomas Bill (S. 1509) and its companion measure, the Barden Bill (H.R. 3846), generally met the approval of educators in their broad provisions, especially the fact that educational benefits were to be administered on a national level by the U. S. Office of Education. Some

essential revisions of the two measures were needed, however, and recommendations to this effect were made on January 10, 1944, by representatives of 21 national associations of education. Before any serious consideration could be given to these recommendations, attention was focused in Washington on a substitute measure, the American Legion Omnibus Bill (S. 1617). This new bill appeared to be receiving much support in Washington, but was generally condemned by educators because of the fact that the educational program would be administered by the Veterans Administration. The Veterans Administration is almost exclusively concerned with vocational rehabilitation. As a result, educational agencies urged a compromise between the Thomas Bill and the Omnibus Bill. On March 13, a new Omnibus Bill (S. 1767) was introduced into the Senate; this incorporated many of the provisions of the Thomas Bill, and it supplanted both S. 1509 and S. 1617. In the House of Representatives, the Rankin Bill (H.R. 4357) was introduced on March 8. It is a companion bill to S. 1767 and replaced H.R. 3846, and the original Rankin Bill with the same number.

On March 13, 1944, the American Council on Education again sponsored a conference of the 21 educational associations to consider the situation brought about by the new Omnibus Bill. At this conference, the following statement of principles was unanimously adopted as being essential to any legislation which would provide educational benefits for discharged military personnel.

1. That the education of veterans under this act should be administered through the authorized educational agencies, federal, state, and local.

2. That responsibility for certification of eligibility of the individual in terms of military service and subsistence payments to individuals should rest with the Veterans Administration.

3. That in each state there shall be designated or created a duly authorized state educational agency which shall be broadly representative of the various levels and types of education within the state. The functions of such a state educational agency should be:

a. To furnish lists of approved educational or training institutions within the state.

b. To advise and assist the approved educational or training institutions furnishing training under this act.

c. To determine, subject to policies to be established on a national basis, the amount of payments to the educational or training institutions furnishing training under this act.

d. To provide educational and vocational guidance.

4. That the educational or training institution should determined the qualifications of the individual for study in such institution and for continuance in the courses.

5. That the individual should be free to select the institution in which he wishes to study, and, after counseling, to select the program of study which he desires to pursue.

It is generally believed in Washington that the termination of hostilities will not involve any great rush of veterans to college and university campuses. Such a point of view follows from such considerations as the following.

1. It is planned to demobilize men upon a gradual basis.

2. It is generally believed that a large number of younger men will be kept in uniform for a considerable period of time to be used for policing duties.

3. It is also believed by some persons that the drafting of 18-year-olds will be continued for a period of time. In this connection, two bills (Wadsworth H.R. 1906 and May H.R. 3947), providing for compulsory military training, have been introduced into Congress; these measures provide that their terms are to be placed in effect six months after the present war. The May Bill requires that the year of compulsory service shall begin at age 17, or at the completion of high school; the Wadsworth Bill leaves a three-year period, 18–21, in which the individual may select the year.

Educational measures for veterans being considered in Washington appear to be premised upon the assumption that approximately a million men will elect to continue their higher education after demobilization. A consultant to the U. S. Office of Education has estimated, however, that only about 150,000 will be able to qualify for a standard four-year college course. Great emphasis is being placed upon elaborate vocational rehabilitation measures.

THE MATHEMATICAL ASSOCIATION OF AMERICA

THE ANNUAL MEETING OF THE MICHIGAN SECTION

The annual meeting of the Michigan Section of the Mathematical Association of America was held Saturday, March 27, 1943, at the University of Michigan in Ann Arbor. This meeting also constituted the meeting of the mathematics section of the Michigan Academy of Science, Arts, and Letters. Morning and afternoon sessions were held, in addition to a luncheon-business meeting at noon. Professor Wayne Dancer of the University of Toledo, Chairman of the Section, presided at all the meetings.

Forty-three persons were in attendance, including the following thirty-three members of the Association: N. H. Anning, W. D. Baten, J. W. Bradshaw, J. B. Brandeberry, Angeline J. Brandt, Jack Britton, R. V. Churchill, C. J. Coe, A. H. Copeland, C. C. Craig, Wayne Dancer, P. S. Dwyer, Peter Field, C. H. Fischer, V. G. Grove, T. H. Hildebrandt, L. A. Hopkins, L. G. Johnson, L. S. Johnston, P. S. Jones, Wilfred Kaplan, L. C. Karpinski, C. E. Love, E. A. Nordhaus, J. K. Peterson, G. Y. Rainich, E. D. Rainville, L. J. Rouse, T. R. Running, B. M. Stewart, R. M. Thrall, E. T. Welmers, R. L. Wilder.

At the business meeting Professor G. C. Speeker of Michigan State College was elected Chairman, and Professor C. J. Coe of the University of Michigan was reelected Secretary-Treasurer for the ensuing year. The Secretary was directed to prepare a statement expressing the sorrow of the Section at the death of Professor E. R. Hedrick, and to send this statement to the Secretary of the Association for forwarding to the bereaved family.

The following program was presented:

1. *A Significance test for an index*, by Professor C. C. Craig, University of Michigan.

In a certain biological investigation, samples of N_1 values of x_1, N_2 values of x_2, and N_3 values of x_3, were supposed to be drawn from three normal universes with means α_1, α_2, and α_3, respectively, and with a common variance σ^2. The index $(\bar{x}_1-\bar{x}_3)/(\bar{x}_1-\bar{x}_2)=k$ was formed ($\bar{x}_1$, $\bar{x}_2$, and $\bar{x}_3$ being the means of the corresponding samples), and it was desired to test the hypothesis that $(\alpha_1-\alpha_3)/(\alpha_1-\alpha_2)=\kappa$. Professor Craig pointed out that this amounts to the test of the linear hypothesis

$$\alpha_1 (1 - \kappa) + \alpha_2 - \alpha_3 = 0,$$

for which the Student-Fisher t-test is at once available. The test in this case was set up and numerically illustrated, and confidence limits for κ were calculated.

2. *A tribute to Earl Raymond Hedrick*, by Professor L. S. Johnston, University of Detroit.

Professor Johnston drew from his intimate personal relations with Professor Hedrick to sketch some details of the personality of that eminent mathematician.

3. *A simplified treatment of Riemann integration*, by Professor A. H. Copeland, University of Michigan.

The speaker presented a new definition of the Riemann integral, and an existence proof for it. The concept of uniform continuity of the integrand was not required. He showed how the definition is applied to physical problems, and brought out the fact that this treatment is both simpler and more rigorous than that usually offered.

4. *Reduction of differential equations to linear form*, by Professor E. A. Nordhaus, Michigan State College.

This paper dealt with the conditions under which an ordinary differential equation is reducible to linear form by a transformation of the dependent variable. An explicit expression for the transformation was given. This was then applied to first order equations to obtain a solution of the Bernoulli equation.

5. *Playing with a Pascal square*, by Professor W. C. Rufus, University of Michigan.

Due to the illness of Professor Rufus, his paper was presented by Professor

N. H. Anning. Professor Rufus had arranged the numbers of the Pascal triangle in a square, and found simple expressions for the numbers in any row, column, or diagonal, and for the ratio of any number to another at any given distance and direction from it. Application was made to expansions of a binomial and to the calculation of probabilities.

6. *A maximum problem*, by Dr. B. M. Stewart, Michigan State College.

Dr. Stewart's paper was published in this MONTHLY, vol. 49, 1942, pp. 454–456.

7. *Constrained vectors in the plane*, by L. G. Johnson, University of Michigan.

This speaker discussed the addition of vectors each of which is constrained to remain on a fixed line. In case two such lines intersected, he placed the terminal point of the first vector and the initial point of the second at the intersection, and defined the sum as the constrained vector running from the initial point of the first to the terminal point of the second. It was shown that this non-commutative addition enjoys certain of the usual properties of addition. Several interesting applications were made.

8. *Analyzing degrees of freedom into comparisons when the classes do not contain the same number of items*, by Professor W. D. Baten, Michigan State College.

Professor Baten showed how to analyze the degrees of freedom in analyses of variances when the classes do not contain the same number of observations. The "treatment sum of squares" was broken up into sums of squares with single degrees of freedom. Two proofs were offered, one based upon orthogonal linear transformations, and the other based upon the fact that a certain combination of quadratic forms each with a single degree of freedom is equivalent to a certain other quadratic form which is the sum of the squares pertaining to "treatment."

9. *New methods in air navigation*, by Professor H. C. Carver, University of Michigan, introduced by the Secretary.

The speaker outlined the basic principles of the graphical method of dead reckoning employed in air navigation, and applied them to problems of interception and radius of action. He also gave a brief discussion of numerical dead reckoning, in which he has developed new methods.

10. *A model of the celestial sphere*, by Professor L. S. Johnston, University of Detroit.

Professor Johnston exhibited a nicely constructed model of the celestial sphere, the various great and small circles of which were represented by circles of wire welded together. The model was about two feet in diameter. He gave some practical suggestions on the construction of such models and their use in the teaching of classes in spherical trigonometry and navigation.

C. J. COE, *Secretary*

THE MARCH MEETING OF THE SOUTHERN CALIFORNIA SECTION

The twenty-fourth regular meeting of the Southern California Section of the Mathematical Association of America was held at Los Angeles City College on Saturday, March 11, 1944. Professor D. C. Duncan, Chairman of the Section, presided at the morning and afternoon sessions.

The attendance was sixty-five, including the following thirty-five members of the Association: O. W. Albert, Harry Bateman, E. T. Bell, L. T. Black, Sister Myrtie Collier, P. H. Daus, R. P. Dilworth, D. C. Duncan, Harriet E. Glazier, Sister Miriam Hand, E. J. Hills, Frances C. Hinds, G. H. Hunt, C. G. Jaeger, Glenn James, E. M. Justin, G. R. Kaelin, G. R. Livingston, Ada A. McClellan, G. F. McEwen, P. M. Niersbach, W. B. Orange, W. T. Puckett, H. R. Pyle, E. C. Rex, J. M. Robb, G. E. F. Sherwood, R. H. Sorgenfrey, A. E. Taylor, B. P. Taylor, W. I. Thompson, V. C. Throckmorton, Morgan Ward, W. M. Whyburn, M. A. Zorn.

At the business meeting the following officers were elected for the next year: Chairman, P. G. Hoel, University of California at Los Angeles; Vice-Chairman, R. P. Dilworth, California Institute of Technology; Program Committee, Frances C. Hinds, Chairman, G. R. Kaelin, William Glenn, and the Secretary. It was decided to hold the next meeting at George Pepperdine College in Los Angeles on March 10, 1945.

The following papers were presented:

1. *Diffusion from a circular area having an initial concentration exceeding that outside by a constant*, by Professor G. F. McEwen, Scripps Institution of Oceanography.

This paper dealt with the problem of finding a solution, in a form suitable for numerical application, of the diffusion equation

$$\frac{\partial u}{\partial t} = a^2\left(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2}\right) \quad \text{or} \quad \frac{\partial u}{\partial t} = a^2\left(\frac{\partial^2 u}{\partial r^2} + \frac{1}{r}\frac{\partial u}{\partial r}\right)$$

subject to the boundary conditions

$$u = c \quad \text{for} \quad r < r_1, \quad \text{and} \quad u = 0 \quad \text{for} \quad r > r_1, \quad \text{when} \quad t = 0.$$

The formal solution was obtained in terms of a definite integral involving Bessel functions. Except in the particular cases corresponding to the center and circumference, it was necessary first to differentiate the integral, and to resort to approximate methods of integration, in order to obtain close approximation in terms of tabulated functions.

2. *Concerning the "width" of plane continua*, by Dr. R. H. Sorgenfrey, University of California at Los Angeles.

By use of the concept of congruent continua, there can be assigned to each plane continuum a non-negative number called its width. The measure of a continuum is not greater than the product of its diameter and its width. If a con-

tinuum has width zero, it can contain no triod nor can it separate the plane; it must therefore be irreducible between some two of its points. In order for there to exist in some bounded portion of the plane infinitely many disjoint continua congruent to a given continuum M, it is necessary and sufficient that M have width zero. An example is given of a non-locally connected continuum having width zero.

3. *The inversion of order of doubly infinite integrals*, by Professor Morgan Ward, California Institute of Technology.

Professor Ward discussed criteria for the integration of series term by term, and for the inversion of order of integration in iterated integrals in which both ranges of integration are infinite. The criteria involved no reference to uniform convergence, but to absolute convergence instead. The intent was to furnish easily applicable tests for use in such problems as occur frequently in the analysis employed in mathematical physics.

4. *An application of the Hartmann-Cornu formula to the determination of radial velocities*, by Professor E. M. Justin, University of California at Los Angeles.

The speaker gave a brief description of the telescope, spectrograph, and measuring engine, and employed slides for purposes of illustration. He then discussed the reduction formula by means of which wave lengths, and finally stellar velocities, are obtained from the measured lines on the sectral flats. A few remarks were made about stellar velocities in general along the line of sight.

5. *Some applications of solid analytic geometry to aircraft problems*, by J. J. Apalategui, Douglas Aircraft Company, introduced by L. J. Adams.

Mr. Apalategui remarked that solid analytic geometry can well be termed "mass production mathematics" as contrasted with the more common trigonometric and descriptive geometry methods of solving for the dimensions which define aircraft parts. Mass production of airplanes calls for correct detail parts which go together without rework in the final assembly of the plane. Some applications of solid analytic geometry to this problem were itemized as follows: the equations of basic wing and empennage lines used to determine the shapes of both normal and canted structural parts; determination of many true and projected angles which define the angularity of structural units; calculation of distances between points, lines, and planes; and the equations for the rotation of axes, which are used in the positioning of parts in any convenient subassembly position in the detail breakdown of an airplane. The speaker distributed in mimeographed form a detailed description of the methods used, complete with diagrams and tables.

6. *The engineer and his mathematics*, by Dr. Adam Zaborski, Lockheed Aircraft Corporation, introduced by Professor Clifford Bell.

The speaker observed that the engineer is interested in mathematics as a tool rather than as an abstract science. He stated that in order to use mathematics

effectively, the engineer should possess three qualifications, namely: he should know how to express physical laws in mathematical form; he should know how to perform mathematical operations according to set rules; he should be able to deduce the physical interpretation of mathematical results. Mr. Zaborski considered that the first and last of these qualifications are the most important, and that they are also the most neglected in our engineering schools. He stressed the need for instructors who can lead engineering students to appreciate the usefulness of certain phases of advanced mathematics such as the calculus of variations, the calculus of finite differences, operational calculus, integral equations, matrices, and projective geometry.

7. *The analytic viewpoint applied to certain aircraft problems*, by R. A. Liming, North American Aviation, introduced by Professor P. H. Daus.

Mr. Liming emphasized the necessity for functional organization within the factory to secure dimensional coordination in all departments which are concerned in any manner with the dimensions of the airplane. He explained how a master dimension group, using analytic methods, can obtain this desired result. The methods used to obtain this coordination in the building of the North American Mustang were discussed. He made a plea for students better trained to think analytically in geometric situations.

P. H. DAUS, *Secretary*

CALENDAR OF FUTURE MEETINGS

Twenty-Seventh Summer Meeting, Wellesley, Mass., August 12–14, 1944.
Twenty-Eighth Annual Meeting, Chicago, Ill., November 24–26, 1944.

The following is a list of the Sections of the Association with dates of future meetings so far as they have been reported to the Secretary.

ALLEGHENY MOUNTAIN
ILLINOIS
INDIANA, Indianapolis, November 10, 1944
IOWA
KANSAS
KENTUCKY
LOUISIANA-MISSISSIPPI
MARYLAND-DISTRICT OF COLUMBIA-VIRGINIA
METROPOLITAN NEW YORK
MICHIGAN
MINNESOTA
MISSOURI
NEBRASKA
NORTHERN CALIFORNIA, San Francisco, January 27, 1945
OHIO
OKLAHOMA
PHILADELPHIA, Philadelphia, November, 1944
ROCKY MOUNTAIN
SOUTHEASTERN
SOUTHERN CALIFORNIA, Los Angeles, March 10, 1945
SOUTHWESTERN
TEXAS
UPPER NEW YORK STATE
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The Chauvenet Prize

In the year 1925, the MATHEMATICAL ASSOCIATION OF AMERICA established a prize of one hundred dollars for the best expository paper published in English during successive periods of five years by a member of the Association. Through two subsequent gifts the prize is now awarded every three years. The last award was made in December 1941 to Professor Saunders Mac Lane for his two papers in the *American Mathematical Monthly:* "Modular fields," volume 47, 1940, pp. 259-274, and "Some recent advances in algebra," volume 46, 1939, pp. 3-19.

As determined more recently by the Trustees, the prize is to be fifty dollars and is to be awarded for a noteworthy expository paper such as will come within the range of profitable reading of members of the Association. The purpose of the prize is to stimulate expository contributions in mathematical journals on the part of the younger American scholars. The award does not apply to books, although the CARUS MONOGRAPHS are expository in character and on this score might be included; they carry their own reward.

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